ADVANCES IN MATHEMATICS 19, 48–105 (1976)

Elliptic (Second-Order) Partial Differential Equations with Measurable Cœfficients and Approximating Integral Equations

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INTRODUCTION

1. This paper deals with linear elliptic second-order partial differential equations with bounded real-valued measurable coefficients. We emphasize that we are concerned with equations in nondivergence form and that no smoothness assumptions are made on the coefficients. For the sake of simplicity we consider leading terms only. Thus, we are concerned with equations of the form

$$\sum_{i,k=1}^{m} a_{ik}(x)\, u_{x_i x_k} = f(x), \tag{1}$$

where the a_{ik} are real-valued functions in $L^\infty(R^m)$, $a_{ik} = a_{ki}$ and the ellipticity condition

$$\mu \leqslant \sum_{i,k=1}^{m} a_{ik}(x)\, \xi_i \xi_k / (\xi_1^2 + \cdots + \xi_m^2) \leqslant M \tag{2}$$

(where $\mu, M =$ positive constants), is satisfied, i.e., the (symmetric) matrix $a(x) = (a_{ik}(x))_{i,k=1,\ldots,m}$ has its eigenvalues in some compact subset of $]0, +\infty[$, which is independent of x.

We are mainly interested in Dirichlet problems, i.e., in solutions of (1) which are defined on a given open set $G \subset R^m$ and assume prescribed values on the boundary ∂G.

2. Very satisfactory results (on existence and uniqueness of solutions of Dirichlet and other boundary value problems, on local properties of solutions, on local or global regularization, etc.) are known for linear elliptic second-order equations with measurable coefficients *in two variables* ($m = 2$), the theory of which probably began with the works of Bers and Nirenberg [7] and Nirenberg [46, 47]. In [46], a tool was introduced, which seems to be the key to such equations: the so-called Bernstein inequality. Indeed, the most important and typical results (i.e., the strongest a priori estimates of solutions) for elliptic equations in two variables with nonsmooth coefficients have been deduced by many authors from this inequality. We mention, for example, the following result (see [63]). If f is square-integrable in a sufficiently smooth two-dimensional domain G, then there exists exactly one solution u of Eq. (1) ($m = 2$), vanishing on the boundary and belonging to the Sobolev space $W^{2,2}(G)$. As is well known, $W^{r,p}(G)$ is the collection of all functions $u \in L^p(G)$, endowed with $L^p(G)$-derivatives up to the order r.

Contributions to linear elliptic equations in two variables with measurable coefficients are in [6, 9, 10, 18, 20, 27, 31, 37, 38, 51, 52, 58, 63, 65, 66]. See also [30, 44].

Unfortunately, the situation for elliptic second-order equations with nonsmooth coefficients *in more than two variables* ($m \geqslant 3$) is completely different.

Some results valid for $m = 2$ have been extended to the case $m \geqslant 3$, but only for a very special class of equations introduced by Cordes [15]. An equation of the form (1) is of the Cordes type if the eigenvalues of the matrix (a_{ik}) are not too dispersed; more precisely, if

$$\left(\sum_{k=1}^{m} a_{kk}(x)\right)^2 \geqslant (m - 1 + \epsilon) \sum_{i,k=1}^{m} a_{ik}(x)^2, \qquad (0 < \epsilon < 1). \tag{3}$$

As is easy to see, every elliptic equation in two variables is of the Cordes type, while in $m \geqslant 3$ variables, the Cordes condition is a special one. Equations of the Cordes type have been studied in [12–14, 21, 25, 28, 32, 61, 64]. In [64], the following result is proved. Under the hypotheses (2) and (3), for every $f \in L^2(G)$, the Dirichlet problem for (1) with homogeneous boundary conditions has exactly one solution u in $W^{2,2}(G)$, provided that the boundary of G (is sufficiently smooth and) has mean curvature of constant sign.

Simple arguments, based on counterexamples (see [64]) and on the closed graph theorem, show that, if $m \geqslant 3$ and if the Cordes hypothesis (3) is dropped, the Dirichlet problem for (1) does not necessarily have solutions in $W^{2,2}(G)$, even if the ellipticity condition (2) is retained and the right-hand side f is in $L^2(G)$, the domain G and the boundary data being arbitrarily smooth.

The Sobolev space $W^{2,2}$ is not suitable, then, for our consideration of (1) with $m \geqslant 3$, provided we wish to impose only the ellipticity hypothesis (2) with no additional assumptions on the coefficients.

On the other hand, the inadequacy of an L^2-theory is consistent with a more general situation. Roughly speaking, if a priori estimates of solutions belonging to some Sobolev space $W^{2,p}$ hold in terms of only the L^p-norm of the right-side f, the ellipticity constants μ, M (2), and a suitable norm of the boundary data, then the exponent p cannot leave a certain range, depending on the ellipticity constants. Useful tools in the study of (among other things) a priori estimates depending on ellipticity constants only, are the maximizing–minimizing operators introduced in [54] and also considered by Miller [40, 43] and Oddson [48]. Thus, in [54], it is shown that an inequality of the form

$$\| u \|_{L^p(G)} \leqslant (\text{const depending on } \mu, M \text{ only}) \| f \|_{L^p(G)}, \tag{4}$$

where G is a ball and u is a $C^2(\bar{G})$-solution of (1) vanishing on the boundary, is impossible if $1/p \geqslant 1/m + (1 - (1/m))(\mu/M)$. In the same paper, it is proved that if G is a ball and $m/2 < p < m$, the maximum of a $W^{2,p}(G)$-solution of (1) vanishing on the boundary cannot be estimated by the $L^p(G)$-norm of f (without some smoothness assumptions on the coefficients) if the ellipticity constant μ is too small, i.e., if $(\mu/M) < ((m/p) - 1)/(m - 1)$.

Some results seem to indicate that the Sobolev space $W^{2,p}$, with $p = m$ (the number of dimensions), is a somewhat privileged class for solutions of elliptic second-order equations with nonsmooth coefficients. In fact, Aleksandrov [2–5] and Pucci [49] prove that *the maximum of solutions* (with *m-integrable second derivatives* on an m-dimensional domain and verifying homogeneous Dirichlet boundary conditions) of any elliptic second-order equation with bounded measurable coefficients *can be estimated by the L^m-norm of the right-hand side.* More precisely, given a solution u of (1) (or even of a more general equation, where lower-order terms are added), which is in the class $W^{2,m}$ on some open bounded set $G \subset R^m$ and continuous up to and vanishing on the boundary, then

$$\max_{x \in \bar{G}} | u(x) | \leqslant (\text{const}) \| f \|_{L^m(G)}, \tag{5}$$

where the constant depends only on the ellipticity constants μ, M and the geometry of G.

The a priori inequality (5) obviously yields uniqueness of solutions of class $W^{2,m}$ of Dirichlet problems for elliptic equations with measurable coefficients. As pointed out in [49], (5) contains a strong form of the maximum principle for solutions in $W^{2,m}$.

Formula (5) also implies the existence of something that should be called a Green's function. In fact, via the Riesz theorem on the representation of bounded linear functionals on Lebesgue spaces, we easily infer from (5) that there exists a map g (at least one) from $G \times G$ into the reals such that $g(x, \cdot) \in L^{m'}(G)$ for every fixed x ($m' = m/m - 1$) and

$$u(x) = -\int_G g(x, y) f(y) \, dy, \tag{6}$$

where u is the (unique) solution of (1) in the class $W^{2,m}(G)$ and vanishing on the boundary. The existence of such a g was also remarked by Krylov [29]. If the coefficients of Eq. (1) are smooth, then the g above is exactly the classical Green's function. If the coefficients of (1) are merely measurable, we could expect the properties of g to differ substantially from those of Green's functions for smooth equations. This feeling is suggested by the work of Miller [39] on the Poisson kernel and his papers [40, 41], and that of Landis [33] on regular boundary points.

Incidentally, the boundary behavior of solutions with Hölder-continuous Dirichlet boundary data of nonsmooth elliptic second-order equations is settled in [40, 53].

As is well known, a priori estimates provide the principal tools for proving existence of solutions of partial differential equations. In other words, inequalities estimating a priori the most significant norms of the relevant solutions in terms of the data are needed. For instance, if we want to prove the existence of a solution to Eq. (1), belonging to some Sobolev class $W^{2,p}$ on a domain G and verifying some boundary conditions (e.g., Dirichlet conditions), we need an a priori inequality of the form

$$\sum_{i,k=1}^{m} \| u_{x_i x_k} \|_{L^p(G)} \leqslant (\text{const})(\| f \|_{L^p(G)} + \text{a norm of boundary data}), \tag{7}$$

where the constant does not depend on u or the smoothness of the coefficients. From our point of view, the constant is permitted to depend on the coefficients a_{ik} through the ellipticity constants μ, M (2) only.

Formula (7) holds with $p = 2$ if the Cordes condition (3) is verified; in particular, (7) holds with $p = 2$ for every elliptic equation if m, the number of dimensions, is 2 (see [13, 14, 63, 64]). On the other hand, there exist elliptic equations with discontinuous coefficients for which (7) holds for any p. For instance, the equations with piecewise constant coefficients considered by Lorenzi [34]. We refer to [35, 36] for an interesting example of an elliptic equation with discontinuous coefficients, such that (7) holds only for special values of p depending on the values of the ellipticity constants.

Roughly speaking, the estimate (7) does not hold in general if the number of dimensions exceeds 2; *an exponent p, such that* (7) *holds for every elliptic equation in $m \geqslant 3$ variables does not exist.* In other words for every p $(1 \leqslant p < +\infty)$ and every $m \geqslant 3$, there exists an elliptic equation of the form (1) and (2), such that (7) does not hold.

If $1 \leqslant p < m$, an instance of such an equation is the one described in [54] in the discussion of the inequality (4). If $p \geqslant m$, an ad hoc example has been constructed by Uralt'seva [67].

3. The concluding remarks of the previous paragraph show essentially that no Sobolev space $W^{2,p}$ contains all solutions of elliptic second-order equations with nonsmooth coefficients.

This fact suggests that it would be worthwhile to consider wider classes of solutions than Sobolev spaces $W^{2,p}$, e.g., classes of solutions unendowed with derivatives, but verifying elliptic differential equations in a suitable generalized sense.

One of the aims of this paper is a tentative definition of such generalized solutions. Section 2 is devoted to this matter. In Section 3, we prove a maximum principle for these solutions.

Dirichlet problems are considered in Section 4. We show that solutions (generalized or not) of Dirichlet problems can be approximated by means of solutions of certain integral equations. These integral equations are suggested by our definition of generalized solutions; they simulate (in some sense) the differential equation and can be actually solved by iterative procedures. As pointed out in Section 5, this treatment of Dirichlet's problems resembles the classical Monte Carlo method for the Laplace equation.

Section 1 has an introductory character and contains no decisive results: We explain our ideas in the case of the simplest elliptic equation, i.e., the Laplace equation.

1. Introductory Remarks, the Equation $\Delta u - \lambda u = f$ in R^m

1.1. In this section we prepare the way by remarking on equations of the form

$$(E - \lambda)u = f, \tag{1.1}$$

where E is the strong derivative at $t = 0$ of a function $0 \leqslant t \to A(t)$, with values in the ring of bounded linear operators on a Banach space X, λ is a real number and f is a given element in the range of $E - \lambda$. The derivative E is defined as

$$Eu = \text{strong limit as } t \downarrow 0 \text{ of } (A(t)u - A(0)u)/t \tag{1.2}$$

and the domain of E consists of all vectors $u \in X$ for which this limit exists.

We suppose that $A(0) = 1$ (*the identity operator*), $A(t)$ *is nonexpansive for every* $t > 0$, i.e.,

$$\| A(t)u \| \leqslant \| u \|, \tag{1.3}$$

λ *is positive.*

We want to show the convergence of a type of finite-difference method, namely, that any solution u of (1.1) is the limit (as t approaches zero) of a solution $v(t)$ of the equation obtained from (1.1) by replacing E with the differential quotient $(A(t) - 1)/t$. More precisely, we show:

(i) For every $t > 0$, there exists a unique solution $v(t)$ of the equation

$$(((A(t) - 1)/t) - \lambda)\, v(t) = f, \tag{1.4}$$

(ii) The estimate: $\| v(t)\| \leqslant (1/\lambda) \| f \|$ holds.

(iii) $v(t)$ converges strongly as $t \downarrow 0$ to any solution u of Eq. (1.1).

Thus, in particular, *the operator* $E - \lambda$ *is one-to-one* and $\|(E - \lambda)^{-1}f\| \leqslant (1/\lambda) \| f \|$ for every f in the range of $E - \lambda$. This remark will be useful later.

In fact,

$$((A(t) - 1)/t) - \lambda = -((1 + \lambda t)/t)[1 - (1 + \lambda t)^{-1} A(t)],$$

where $\|(1 + \lambda t)^{-1} A(t)\| \leqslant 1/(1 + \lambda t) =$ a number strictly less than 1, the inequality being an obvious consequence of (1.3). Then, the inverse operator of $((A(t) - 1)/t) - \lambda$ can be expanded in a Neumann series

$$\left[\frac{A(t) - 1}{t} - \lambda\right]^{-1} = -\frac{t}{1 + \lambda t} \sum_{n=0}^{+\infty} (1 + \lambda t)^{-n} A(t)^n \tag{1.5}$$

absolutely convergent in the uniform operator topology, and

$$\left\|\left[\frac{A(t) - 1}{t} - \lambda\right]^{-1}\right\| \leqslant \frac{t}{1 + \lambda t} \sum_{n=0}^{+\infty} (1 + \lambda t)^{-n} = \frac{1}{\lambda}. \tag{1.6}$$

Thus, we have proved (i) and (ii). To prove (iii), we note that, as follows clearly from (1.1) and (1.4), the difference $v(t) - u$ is a solution of the equation

$$(((A(t) - 1)/t) - \lambda)(v(t) - u) = Eu - ((A(t)u - u)/t).$$

Applying the estimate (1.6), we get

$$\| v(t) - u \| \leqslant (1/\lambda) \| Eu - ((A(t)u - u)/t)\|,$$

where the right side tends to zero as $t \downarrow 0$ by the definition (1.2) of E.

The previous arguments become more transparent if we suppose that the family $\{A(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup of contractions, i.e.,

(i) $A(s + t) = A(s)\, A(t)$, $A(0) = 1$;
(ii) $\| A(t)\| \leqslant 1$;
(iii) $\| A(t)u - u \| \to 0$, for every u if $t \downarrow 0$.

For under this hypothesis, E, the infinitesimal generator of the semigroup, is a densely defined closed operator whose spectrum lies in the west half-plane and the resolvent operator $(z - E)^{-1}$ coincides in the east half-plane with the Laplace transform of the semigroup. In other words, Eq. (1.1), with a positive λ, has a unique solution u for every f in the whole space X and

$$u = -\int_0^{+\infty} e^{-\lambda t} A(t) f \, dt. \tag{1.7}$$

Here, the vector-valued function $0 \leqslant t \to A(t)f$ is a strongly continuous bounded solution (a strongly differentiable bounded solution, if f lies in the domain of E) of the evolution differential equation

$$((d/dt) - E)\cdot = 0,$$

with f as its value at the initial point $t = 0$.

On the other hand, the previous property (iii), concerning the convergence of $v(t)$ to the solution u of (1.1), follows at once from the fact that $v(t)$ behaves for $t \downarrow 0$ like

$$-t \sum_{n=0}^{+\infty} e^{-n\lambda t} A(nt) f,$$

which is a Riemann sum associated with the integral in (1.7). As a matter of fact, the Neuman series (1.5) can be written as

$$\left(\frac{A(t) - 1}{t} - \lambda\right)^{-1} = -\frac{t}{1 + \lambda t} \sum_{n=0}^{+\infty} (1 + \lambda t)^{-n} A(nt),$$

by the semigroup property. Then:

$$\left\| -t \sum_{n=0}^{+\infty} e^{-n\lambda t} A(nt) f - (1 + \lambda t)\, v(t) \right\|$$

$$= \left\| t \sum_{n=0}^{+\infty} (-e^{-n\lambda t} + (1 + \lambda t)^{-n})\, A(nt) f \right\|$$

$$\leqslant t \, \|f\| \sum_{n=0}^{+\infty} ((1 + \lambda t)^{-n} - e^{-n\lambda t})$$

$$= \|f\| \, (t + (1/\lambda) - (t/(1 - e^{-\lambda t}))) \leqslant t \, \|f\|.$$

1.2. In this section, we consider some operator-valued functions whose derivative at a point is the Laplace operator

$$\Delta = \partial^2/\partial x_1{}^2 + \cdots + \partial^2/\partial x_m{}^2.$$

Here, we consider Δ as an unbounded operator acting in $L^p(R^m)$, with domain $= W^{2,p}(R^m)$; p is any number such that $1 < p < +\infty$.

We introduce a measurable real-valued function φ of a scalar variable $r > 0$ such that

$$\begin{aligned} &\int_0^{+\infty} \varphi(r)\, r^{m-1}\, dr = 1/\omega_m \equiv \Gamma(m/2)/(2\pi^{m/2}), \\ &(\omega_m/2m) \int_0^{+\infty} r^2 \varphi(r)\, r^{m-1}\, dr \equiv C < \infty, \end{aligned} \tag{1.8}$$

and consider the convolutions

$$\varphi_t * f,$$

whose kernels are as follows:

$$\varphi_t(x) = t^{-m/2} \varphi(t^{-1/2} \mid x \mid), \qquad t > 0 \tag{1.9}$$

Also, we assume for convenience that

$$\varphi(r) \geqslant 0.$$

Young's inequality gives

$$\| \varphi_t * f \|_{L^p(R^m)} \leqslant \| f \|_{L^p(R^m)} .$$

Moreover, a standard theorem on convolutions guarantees

$$\| \varphi_t * f - f \|_{L^p(R^m)} \to 0 \qquad \text{as} \quad t \downarrow 0$$

for every $f \in L^p(R^m)$; so that the convolution operators $L^p(R^m) \ni f \to \varphi_t * f \in L^p(R^m)$ are contractions (or nonexpansive operators) and converge strongly to the identity operator as $t \downarrow 0$.

We claim:

(i) $\| \varphi_t * f - f \|_{L^p(R^m)} \leqslant Ct \| \Delta f \|_{L^p(R^m)}$, for every f in $W^{2,p}(R^m)$. Here, C is the constant in (1.8).

(ii) if $f \in L^p(R^m)$ is such that $(1/t) \| \varphi_t * f - f \|_{L^p(R^m)}$ is bounded as $t \downarrow 0$, then f is of class $W^{2,p}(R^m)$.

(iii) if f is in $W^{2,p}(R^m)$, then $(1/t)(\varphi_t * f - f)$ tends in $L^p(R^m)$ to $C \Delta f$ as $t \downarrow 0$, where C is the constant in (1.8).

Just for the sake of completeness, we state the estimate

(iv) $\| \varphi_t * f - f \|_{L^p(R^m)} \leqslant t^{1/2} (8mC)^{1/2} \| \mid Df \mid \|_{L^p(R^m)}$, if f is in $W^{1,p}(R^m)$.

Incidentally, (i)–(iv) imply that the kernel (1.9), considered as an approximation in $L^p(R^m)$ of the Dirac mass, is saturated exactly at the order 1 (on saturation theory, see, e.g., [59]). In fact, $\| \varphi_t * f - f \|_{L^p(R^m)}$ tends to zero as t approaches zero if $f \in L^p(R^m)$; it is $O(t^{1/2})$ if $f \in W^{1,p}(R^m)$; and it is $O(t)$ if $f \in W^{2,p}(R^m)$. On the other hand, if $\| \varphi_t * f - f \|_{L^p(R^m)} = o(t)$ as $t \downarrow 0$ for some $f \in L^p(R^m)$, then (ii)–(iii) show that f is harmonic; hence, $f \equiv 0$ (see [62, Theorem 4.1]).

The properties (i)–(iii) mean exactly that *the map*:

$$\begin{aligned} t &\to \text{the convolution operator } \varphi_t * \qquad && \text{if} \quad t > 0 \\ &\to \text{the identity operator} && \text{if} \quad t = 0 \end{aligned} \tag{1.10}$$

is a function with values in the ring of bounded linear operators in $L^p(R^m)$, whose derivative at $t = 0$ is $C\Delta$, a numerical multiple of the Laplacian. We emphasize again that Δ must be understood here as an unbounded operator in $L^p(R^m)$ with domain $= W^{2,p}(R^m)$.

Note in particular, that if we choose

$$\varphi(r) = (4\pi)^{-m/2} \exp(-r^2/4), \tag{1.11}$$

the function (1.10) becomes a strongly continuous semigroup of contractions on $L^p(R^m)$, having Δ as infinitesimal generator. In fact, with the choice (1.11), the convolution

$$(\varphi_t * f)(x) \equiv \int_{R^m} f(y) \exp(-\mid x - y \mid^2/4t)(dy/(4\pi t)^{m/2}) \tag{1.12}$$

can be written

$$(\varphi_t * f)\hat{\ } (\xi) = \exp(-t \mid \xi \mid^2) \quad (\xi)$$

for every function f in some dense subset of $L^p(R^m)$, $f \in C_0^\infty(R^m)$ say; here, $\hat{\ }$ denotes the Fourier transformation:

$$\hat{f}(\xi) = \int_{R^m} f(x)\, e^{-i(x,\xi)}\, dx,$$

and $\exp(-t \mid \xi \mid^2)$ is the Fourier transform of the kernel $\varphi_t(x) = (4\pi t)^{-m/2} \exp(-\mid x \mid^2/4t)$. The formula following (1.12) plainly shows the semigroup property:

$$\varphi_t * (\varphi_s * f) = \varphi_{t+s} * f \qquad (t, s > 0). \tag{1.13}$$

On the other hand, it is easy to see that property (iii) holds here with $C = 1$. Of course, these last remarks rephrase nothing more than classical properties of the Poisson integral for the heat equation.

Proof of (i) *and* (ii). These properties follow at once from the formula

$$\varphi_t * f - f = Ct\psi_t * (\Delta f), \tag{1.14}$$

where f is any function in $W^{2,p}(R^m)$, φ_t is as before, C is the constant in (1.8), and

$$\begin{aligned} \psi_t(x) &= t^{-m/2}\psi(t^{-1/2} \mid x \mid), \\ \psi(r) &= \frac{1}{C}\int_r^{+\infty} \frac{r^{2-m} - s^{2-m}}{m-2}\, \varphi(s)\, s^{m-1}\, ds. \end{aligned} \tag{1.15}$$

Note that

$$\psi(r) \geqslant 0, \qquad \omega_m \int_0^{+\infty} \psi(r)\, r^{m-1}\, dr = 1; \tag{1.16}$$

the latter being a consequence of the straightforward formula

$$\omega_m \int_0^{+\infty} \psi(r)\, r^{m-1}\, dr = (\omega_m/2mC) \int_0^{+\infty} r^2\varphi(r)\, r^{m-1}\, dr.$$

Incidentally, an easy inspection of (1.15) shows that $\psi(r)$ is a solution of the differential equation

$$-C((d/r)^2 + ((m-1)/r)(d/dr))\, \psi(r) + \varphi(r) = 0,$$

with the behavior at the singular point $r = 0$:

$$(m-2)\, \omega_m r^{m-2} C\psi(r) \to 1 \qquad (r \downarrow 0).$$

Equivalently, $\psi(\mid x \mid)$ is a solution of

$$-C\Delta\psi(\mid x \mid) + \varphi(\mid x \mid) = \text{Dirac mass},$$

as one also can deduce from (1.14) (taking Fourier transforms, for instance).

To prove (1.14), we can suppose $f \in C_0^\infty(R^m)$. By Green's formulas,

$$\begin{aligned} &\int_{|x|=r} f(x)\, H_{m-1}(dx) - \omega_m r^{m-1} f(0) \\ &\qquad = r^{m-1} \int_{|x|<r} ((\mid x \mid^{2-m} - r^{2-m})/(m-2))\, \Delta f(x)\, dx, \end{aligned}$$

where H_{m-1} is the $(m-1)$-dimensional measure. Multiplying both members of the previous equation by $t^{-m/2}\varphi(t^{-1/2}r)$, then, integrating over r and making a change of the order of integrations, we obtain

$$\int_0^{+\infty} t^{-m/2}\varphi(t^{-1/2}r)\,dr \int_{|x|=r} f(x)\,H_{m-1}(dx)$$

$$-f(0)\,\omega_m \int_0^{+\infty} t^{-m/2}\varphi(t^{-1/2}r)\,r^{m-1}\,dr$$

$$=\int_{R^m} \Delta f(x)\,dx \int_{|x|}^{+\infty} \frac{|x|^{2-m}-r^{2-m}}{m-2}\,t^{-m/2}\varphi(t^{-1/2}r)\,r^{m-1}\,dr.$$

This equality reads

$$\int_{R^m} f(x)\,t^{-m/2}\varphi(t^{-1/2}|x|)\,dx - f(0)$$

$$= Ct \int_{R^m} \Delta f(x)\,r^{-m/2}\psi(t^{-1/2}|x|)\,dx,$$

or also,

$$(f*\varphi_t)(0)-f(0)=Ct(\Delta f*\psi_t)(0).$$

The last formula is obviously equivalent to (1.14).

Proof of (ii). Let $f\in L^p(R^m)$ be such that

$$\sup_{t>0}(1/t)\,\|\varphi_t*f-f\|_{L^p(R^m)}=A<\infty$$

and let $f_\epsilon=J_\epsilon*f$, be the convolution of f with a mollifier (that is: $J_\epsilon(x)=\epsilon^{-m/2}J(\epsilon^{-1/2}x)$, $J\in C_0^\infty(R^m)$, $J\geqslant 0$, $\int J(x)\,dx=1$). The function f_ϵ is in $C^\infty(R^m)$ and converges to f in $L^p(R^m)$ as $\epsilon\downarrow 0$. Moreover, by Young's inequality: $\|\varphi_t*f_\epsilon-f_\epsilon\|_{L^p(R^m)}=\|J_\epsilon*(\varphi_t*f-f)\|_{L^p(R^m)}\leqslant \|\varphi_t*f-f\|_{L^p(R^m)}$ and so

$$(1/t)\,\|\varphi_t*f_\epsilon-f_\epsilon\|_{L^p(R^m)}\leqslant A \qquad \text{for every}\quad t>0.$$

We let $t\downarrow 0$ in this inequality and obtain in virtue of property (iii), since f_ϵ is smooth,

$$\|\Delta f_\epsilon\|_{L^p(R^m)}\leqslant A/C.$$

Such an estimate implies that

$$\| \partial^2 f_\epsilon / \partial x_i \, \partial x_j \|_{L^p(R^m)} \leqslant \text{const independent of } \epsilon.$$

For one has

$$\partial^2 u / \partial x_i \, \partial x_j = -R_i R_j \Delta u, \tag{1.18a}$$

as can be seen readily by using Fourier transforms, and

$$\| R_i u \|_{L^p(R^m)} \leqslant (\text{const depending on } p \text{ only}) \, \| u \|_{L^p(R^m)}, \tag{1.18b}$$

owing to a theorem of Calderón and Zygmund. Here, u is any smooth function and R_i are the Riesz operators (see [60])

$$R_i u(x) = \text{principal value of } \pi^{-(m+1)/2} \, \Gamma\left(\frac{m+1}{2}\right) \int_{R^m} \frac{x_i - y_i}{| x - y |^{m+1}} f(y) \, dy,$$

that is, the singular integrals whose symbols are

$$-(-1)^{1/2} \, \xi_i / | \xi |.$$

By a theorem on Sobolev spaces, a function f, which is approximated in $L^p(R^m)$ by a family of smooth functions f_ϵ such that all the second derivatives $\partial^2 f_\epsilon / \partial x_i \, \partial x_j$ are uniformly bounded in $L^p(R^m)$, is actually in $W^{2,p}(R^m)$.

The proof of assertions (i)–(iii) is now complete.

1.3. In this section, we describe a procedure for approximating a solution of the equation

$$\Delta u - \lambda u = f, \tag{1.21}$$

where λ is a positive number and f is any given function in $L^p(R^m)$. Here, $1 < p < +\infty$.

This procedure is based on the results of the previous sections.

We consider the solution u of (1.21) belonging to the Sobolev space $W^{2,p}(R^m)$. As is well known, this solution exists (and is unique) whenever f is in $L^p(R^m)$ and the formula

$$u(x) = (K * f)(x) \equiv \int_{R^m} K(x - y) \, f(y) \, dy \tag{1.22}$$

holds, where K is the fundamental solution of the operator $\Delta - \lambda$. K is defined by

$$\hat{K}(\xi) = -1/(\lambda + |\xi|^2), \tag{1.23}$$

$\hat{K}$ being the Fourier transform of K; or

$$K(x) = -(1/2\pi)(2\pi |x| \lambda^{-1/2})^{1-(m/2)} K_{(m/2)-1}(\lambda^{1/2} |x|), \tag{1.24}$$

where

$$K_{(m/2)-1}(r) = \tfrac{1}{2}(r/2)^{(m/2)-1} \int_0^{+\infty} \exp(-t - r^2/4t)\, t^{-m/2}\, dt \tag{1.25}$$

is the modified Bessel function of the second kind of order $(m/2) - 1$.

The procedure we have in mind is explained in the following assertions, which result easily from the content of the previous subsections.

(i) for every $t > 0$, there exists a unique solution $v(t, \cdot) \in L^p(R^m)$ of the integral equation

$$\frac{1}{Ct}\left[\int_{R^m} t^{-m/2}\varphi(t^{-1/2} |x - y|)\, v(t, y)\, dy - v(t, x)\right] - \lambda v(t, x) = f(x). \tag{1.26}$$

Here, φ, φ_t,..., etc., are as in Section 1.2 (in particular, C is the value of the integral $(\omega_m/2m) \int_0^{+\infty} r^2\varphi(r) r^{m-1}\, dr$).

(ii) $\| v(t, \cdot)\|_{L^p(R^m)} \leqslant (1/\lambda) \| f \|_{L^p(R^m)}$ for every $t > 0$.

(iii) $v(t, \cdot)$ converges strongly in $L^p(R^m)$ as $t \downarrow 0$ to the solution $u \in W^{2,p}(R^m)$ of the equation $\Delta u - \lambda u = f$. In fact, we have

$$\| v(t, \cdot) - u \|_{L^p(R^m)} \leqslant (1/\lambda) \| \Delta u - ((\varphi_t * u - u)/Ct)\|_{L^p(R^m)} .$$

Equation (1.26) is an integral equation of the convolution type. Thus, assertion (i) can also be deduced from theorems of harmonic analysis. In fact, as is well known and easy to see (via a "continuous version" of Wiener's theorem on absolutely convergent Fourier series), the spectrum of the convolution $L^p(R^m) \ni f \to K * f \in L^p(R^m)$, where K is any kernel in $L^1(R^m)$, is exactly the closure of the set of the values of $\hat{K}$; thus, the integral equation $K * v - \mu v = f$ has a unique solution $v \in L^p(R^m)$ for every $f \in L^p(R^m)$ if and only if μ is neither zero nor a value of $\hat{K}$. In our case,

$$K = \varphi_t, \qquad \mu = 1 + Ct, \qquad |\hat{\varphi}_t(\xi)| \leqslant \int_{R^m} \varphi_t(x)\, dx = 1;$$

so that μ cannot be zero or a value of $\hat{K}$.

In agreement with Section 1.1, the solution $v(t, \cdot)$ of the integral equation (1.26) has the expansion

$$v(t, \cdot) = -\frac{Ct}{1 + C\lambda t}\,[f + \cdots + (1 + C\lambda t)^{-n}\,(\varphi_t * \cdots * \varphi_t) * f + \cdots] \quad (1.27)$$

(where $(\varphi_t * \cdots * \varphi_t)$ are n factors) absolutely convergent in the metric of $L^p(R^m)$. In the case where φ has the Gaussian form (1.11), $C = 1$ and the right side of (1.27) becomes

$$-(t/(1 + \lambda t))[f + \cdots + (1 + \lambda t)^{-n}\,\varphi_{nt} * f + \cdots]$$

by the semigroup property (1.13). Then, assertion (iii) follows from the fact (already remarked) that the sum of such a series converges in $L^p(R^m)$ as $t \downarrow 0$ to the limit of

$$-t(f + \cdots + e^{-\lambda nt}\varphi_{nt} * f + \cdots).$$

The latter limit clearly is the function

$$-\int_0^{+\infty} e^{-\lambda t}(\varphi_t * f)(x)\,dt$$

$$\equiv -\int_0^{+\infty} e^{-\lambda t}\,dt \int_{R^m} (4\pi t)^{-m/2} \exp(-|\,x - y\,|^2/4t)\,f(y)\,dy,$$

i.e., the function

$$\int_{R^m} K(x - y)\,f(y)\,dy,$$

where K is the fundamental solution (1.24)-(1.25).

The assertion (iii) can be strengthened. Assume that the weight φ has some smoothness and decays rapidly at infinity. (Assume, e.g., $\varphi \in C_0^\infty([0, +\infty[)$ and $\varphi(r) \equiv 1$ for every r in a neighborhood of the origin, although a less crude assumption would suffice.) Then,

$$v(t, \cdot) + \frac{Ct}{1 + C\lambda t} f$$

$\equiv v(t, \cdot)$ minus the first term in the Neumann expansion (1.27) belongs to $C^\infty(R^m)$ and converges to u in the metric of $W^{2,p}(R^m)$ as $t \downarrow 0$.

As an easy consequence, if we refine slightly our procedure, replacing f in (1.26) with a mollified $J_t * f$, then the new $v(t, \cdot)$ is actually in $C^\infty(R^m)$ and does converge to u in $W^{2,p}(R^m)$ as $t \downarrow 0$. This follows essentially from the obvious fact that $tJ_t * f$ goes to zero in $W^{2,p}(R^m)$ as $t \downarrow 0$, whenever f is $L^p(R^m)$. Here, $J_t(x) = t^{-m/2} J(t^{-1/2}x)$ and J is a nonnegative infinitely differential compactly supported function.

For the proof of the above assertion, it is sufficient to show that

$$w(t, \cdot) = v(t, \cdot) + (Ct/(1 + C\lambda t))f - \varphi_t * u \tag{1.28}$$

is in $C^\infty(R^m)$ and that

$$\| \Delta w(t, \cdot)\|_{L^p(R^m)} \to 0 \qquad \text{if} \quad t \downarrow 0. \tag{1.29}$$

For, under the present hypotheses on the smoothness of φ, the kernel φ_t is a mollifier, so $\varphi_t * u$ is in $C^\infty(R^m)$ and converges to u in $W^{2,p}(R^m)$ if $t \downarrow 0$, since $u \in W^{2,p}(R^m)$. On the other hand, the convergence in $L^p(R^m)$ of $\Delta w(t, \cdot)$ to zero implies the convergence to zero in $L^p(R^m)$ of all the second derivatives of $w(t, \cdot)$ (see (1.18)).

From Eqs. (1.26) and $f = \Delta u - \lambda u$, we infer that $w(t, \cdot)$ verifies the integral equation

$$\varphi_t * w(t, \cdot) - w(t, \cdot) = Ct\varphi_t * g(t, \cdot), \tag{1.30}$$

where

$$g(t, \cdot) = \Delta u - (1/Ct)(\varphi_t * u - u) + \lambda(v(t, \cdot) - u) + (Ct/(1 + C\lambda t))(f + \lambda v(t, \cdot)),$$

$$\|g(t, \cdot)\|_{L^p(R^m)} \leqslant 2Ct \|f\|_{L^p(R^m)} + \| \Delta u - (1/Ct)(\varphi_t * u - u)\|_{L^p(R^m)} \to 0 \qquad \text{if} \quad t \downarrow 0, \tag{1.31}$$

the inequality being a consequence of the estimates of v.

The two terms $\varphi_t * w(t, \cdot)$, $\varphi_t * g(t, \cdot)$, appearing in (1.30), are in $C^\infty(R^m)$, since both $w(t, \cdot)$ and $g(t, \cdot)$ are in $L^p(R^m)$ and φ_t is a mollifier. Hence, $w(t, \cdot) \in C^\infty(R^m)$, for Eq. (1.30) expresses it as a difference of infinitely differentiable functions.

Taking derivatives, we get from (1.30)

$$\varphi_t * \Delta w(t, \cdot) - \Delta w(t, \cdot) = Ct(\Delta \varphi_t) * g(t, \cdot).$$

Hence, taking Fourier transforms:

$$\Delta w(t, \cdot) = M_t * g(t, \cdot), \tag{1.32}$$

where M_t is the distribution whose Fourier transform is the function

$$\hat{M}_t(\xi) = (Ct \mid \xi \mid^2/(1 - \hat{\varphi}_t(\xi)))\, \hat{\varphi}_t(\xi). \tag{1.33}$$

We shall show that M_t is a function of the form

$$M_t(x) = t^{-m/2} M(t^{-1/2} x), \qquad \text{where} \quad M \in L^1(R^m). \tag{1.34}$$

With (1.34), we deduce, via Young's inequality, that

$$\| M_t * g(t, \cdot)\|_{L^p(R^m)} \leqslant (\text{const independent of } t)\, \| g(t, \cdot)\|_{L^p(R^m)} .$$

Hence, we obtain (1.29), taking into account (1.31) and (1.32).

Actually, M is a function belonging to the space $\mathscr{S}$ of Schwarz, i.e., M is in $C^\infty(R^m)$ and decays at infinity together with all its derivatives faster then any power of $\mid x \mid$.

In fact, $\hat{M}_t(\xi) = C(t^{1/2} \mid \xi \mid)^2\, \Phi(t^{1/2} \mid \xi \mid)/(1 - \Phi(t^{1/2} \mid \xi \mid))$, or

$$\hat{M}(\xi) = (C \mid \xi \mid^2/(1 - \Phi(\mid \xi \mid)))\, \Phi(\mid \xi \mid),$$

where $\Phi(t^{1/2} \mid \xi \mid)$ is the Fourier transform of $\varphi_t(x) = t^{-m/2} \varphi(t^{-1/2} \mid x \mid)$, namely,

$$\Phi(r) = (2\pi)^{m/2} \int_0^{+\infty} (rs)^{1-m/2}\, J_{(m/2)-1}(rs)\, \varphi(s)\, s^{m-1}\, ds,$$

by Bochner's formula on the Fourier transform of spherically symmetric functions. From the inequality

$$|(r/2)^{-\lambda}\, J_\lambda(r)| \leqslant 1/\Gamma(\lambda + 1)$$

($\lambda > -\frac{1}{2}$; r real, the equality holds only if $r = 0$) and condition (1.8), we get

$$\mid \Phi(r) \mid < (2\pi^{m/2}/\Gamma(m/2)) \int_0^{+\infty} \varphi(s)\, s^{m-1}\, ds = \Phi(0) = 1 \qquad \text{if} \quad r > 0.$$

From the powers series expansion for Bessel functions and condition (1.8), again, we have

$$\Phi(r) = 1 - Cr^2 + \cdots + (-1)^n\, r^{2n} \frac{2^{1-n} \pi^{m/2}}{n!\, \Gamma(n + (m/2))} \int_0^{+\infty} s^{2n} \varphi(s)\, s^{m-1}\, ds + \cdots,$$

the series being convergent for every r, since φ has compact support. Thus, the ratio $Cr^2\Phi(r)/(1 - \Phi(r))$ is an analytic even function everywhere regular. Moreover, it decays at infinity together with all its derivatives faster than any power of r, since Φ has a similar property, due to the smoothness of φ. The proof is complete.

2. Generalized Elliptic Differential Operators

2.1. The subject of the present section is closely related to mean value theorems for solutions of partial differential equations. Here, we give a brief account of some of these theorems and we indicate the connection between them and a notion, to be developed in the following subsections, of generalized solution.

As is well known, a harmonic function u on a domain $G \subset R^m$ is a continuous function u such that:

$$u(x) = (m/\omega_m r^m) \int_{|x-y|<r} u(y)\, dy = \text{the arithmetic mean of } u \text{ on the ball with center at } x \text{ and radius } r \tag{2.1}$$

for every $x \in G$ and every $r <$ distance $(x, \partial G)$. If u is a twice continuously differentiable function (not necessarily harmonic), the deviation of $u(x)$ from the mean of u on a ball with center at x is an average on the same ball of the Laplacian Δu: This is a particular case of formula (1.14). As a consequence, we have the rule

$$(1/r^2) \left[(m/\omega_m r^m) \int_{|x-y|<r} u(y)\, dy - u(x)\right] \to (1/2(m+2))\, \Delta u(x), \qquad \text{if} \quad r \downarrow 0. \tag{2.2}$$

Formula (2.2), which is a theorem for twice continuously differentiable functions, suggests a device for defining a generalized Laplacian of nondifferentiable functions. For instance, if u and f are merely continuous functions, one can say that Δu is equal to f in a generalized sense if the quantity at the left of (2.2) tends to $f(x)/2(m+2)$ as $r \downarrow 0$. This is essentially the point of view of Sato [56, 57]. We note that this definition has the advantage, among others, of preserving apparently the classical maximum principle for the Poisson equation.

A characterization of solutions of constant coefficient partial differential equations with the help of mean value properties, generalizing the aforesaid characterization of harmonic functions, was considered by Flatto [19] and Friedman and Littman [22], who proved the following. Let μ be a nonnegative Borel measure with total mass $= 1$, such that $K = \operatorname{spt}\mu$, the support of μ, is contained in the unit sphere of R^m and not contained in any hyperplane. If u is a continuous function on some open set $G \subset R^m$ having the mean value property

$$u(x) = \int_K u(x + ry)\,\mu(dy)$$

for every $x \in G$ and every positive $r <$ distance $(x, \partial G)$, then u is in $C^\infty(G)$ and satisfies the system

$$\sum_{i_1+\cdots+i_m=n} A_{i_1\cdots i_m} \frac{\partial^n u}{\partial x_1^{i_1}\cdots\partial x_m^{i_m}} = 0 \qquad (n = 1, 2, \ldots),$$

where the coefficients are the moments

$$A_{i_1\cdots i_m} = \int_K x_1^{i_1}\cdots x_m^{i_m}\,\mu(dx).$$

Conversely, every infinitely differentiable solution of the last system of differential equations has the aforesaid mean value property.

As far as elliptic equations of the second order with nonconstant coefficients are concerned, mean value theorems are known especially in the case of equations in divergence form (formally self-adjoint equations), a case out of the scope of this paper. For an elliptic operator of the form

$$E = \sum_{i,k=1}^{m} a_{ik}(x)\,\partial^2/\partial x_i\,\partial x_k\,, \tag{2.3}$$

where the coefficients are supposed merely measurable and bounded and the matrix $a(x) = (a_{ik}(x))$ symmetric and positive definite, Fulks [23] proved

$$\begin{aligned}(1/\omega_m)\int_{|y|=1} u(x + ra(x)^{1/2}\,y)\,H_{m-1}(dy) - u(x)&\\ = (r^2/2m)\,Eu(x) + o(r^2),&\end{aligned} \tag{2.4}$$

u being any twice continuously differentiable function. Here, $a(x)^{1/2}$

denotes the positive square root of the matrix $a(x)$ and H_{m-1} the $(m-1)$-dimensional measure.

Formula (2.4) can be transformed into a more convenient one. Multiplying both sides of (2.4) by mr^{m-1} and integrating with respect to r, we obtain (after an obvious change of variables in the m-fold repeated integral)

$$(1/\text{meas}\, \mathscr{E}(x, r)) \int_{\mathscr{E}(x,r)} u(y)\, dy - u(x) = (r^2/2(m+2))\, Eu(x) + o(r^2), \tag{2.5}$$

where $\mathscr{E}(x, r)$ is the ellipsoid

$$\mathscr{E}(x, r) = \{y \in R^m : (a(x)^{-1} (y - x), y - x) < r^2\},$$

with center at x and axes on the eigenvectors of the matrix $a(x)$ and

$$\text{meas}\, \mathscr{E}(x, r) = (\omega_m/m)\, r^m\, (\det a(x))^{1/2}.$$

In the derivation of (2.5), we have used the rule $|\, a(x)^{1/2}\xi\, |^2 = (a(x)\xi, \xi)$; where ξ is any vector in R^m, $a(x)^{1/2}\xi$ is the product of the matrix $a(x)^{1/2}$ by the vector ξ, $(a(x)\xi, \xi) = \sum_{i,k=1}^{m} a_{ik}(x)\, \xi_i \xi_k ,\ldots$.

Note that the boundaries $\partial\mathscr{E}(x, r)$ of the ellipsoids (2.6) are the level surfaces of the Levi function

$$L(x, y) = (1/(m-2)\, \omega_m)(\det a(x))^{-1/2}\, (a(x)^{-1} (y - x), y - x)^{1-m/2},$$

the fundamental solution of the constant coefficient elliptic operator $\sum_{i,k=1}^{m} a_{ik}(x)\, \partial^2/\partial y_i\, \partial y_k$.

Clearly, formula (2.5), which (following Fulks) can be called an approximate mean value theorem, is a generalization of (2.2). We make the following observations on such a formula, which is a typical instance of a more general situation discussed in the sequel.

(i) We have

$$Eu(x) = 2(m+2) \lim_{r \downarrow 0} (1/r^2) \left[-u(x) + (1/\text{meas}\, \mathscr{E}(x, r)) \int_{\mathscr{E}(x,r)} u(y)\, dy\right], \tag{2.7}$$

where E is the elliptic operator (2.3), $\mathscr{E}(x, r)$ is the set defined by (2.6). Equation (2.7) holds for twice continuously differentiable functions u, or even (as could be proved) for locally integrable functions u with locally integrable second derivatives. In the latter case, the limit in (2.7) must be taken in a suitable metric. The rule (2.7) gives a representation of the differential operator E, which does not depend on the coordinate system in R^m.

(ii) We can define the action of the differential operator E on functions unendowed with second derivatives; i.e., we can define (without requiring any smoothness of the coefficients of E) a class of generalized solutions of the equation $Eu = f$. In other words, the rule (2.7) gives a method of introducing an extension of the formal differential expression (2.3): The domain of such an extension will be the collection of all functions u (belonging to some suitable ground space) for which the right-hand side of (2.7) exists (the limit being taken in the relevant metric); the value of E on every u lying in this domain will be given by (2.7).

Notice that the procedure just described is not very different from the definition of generalized elliptic operators considered by Giraud (see [44, par. 25]). Giraud uses in place of (2.7) the formula

$$Eu(x) = \lim_{r\downarrow 0} \frac{2m}{r^2} \left\{ \sum_{k=1}^{m} \frac{1}{2m} [u(x + r(\lambda_k(x))^{1/2} e_k(x)) \right.$$
$$\left. + u(x - r(\lambda_k(x))^{1/2} e_k(x))] - u(x) \right\},$$

which can be considered a discrete version of (2.7). Here, $e_1(x),\dots, e_m(x)$ is an orthonormal set of eigenvectors of the (symmetric) matrix $a(x)$ and $\lambda_k(x)$ is the (positive) eigenvalue associated with $e_k(x)$.

2.2. Here, we describe a method of defining generalized solutions of elliptic second-order differential equations with measurable coefficients, following the ideas sketched in the previous subsection.

Consider an elliptic operator of the form

$$\sum_{i,k=1}^{m} a_{ik}(x)\, \partial^2/\partial x_i\, \partial x_k \,. \tag{2.10}$$

We suppose that the coefficients are measurable and bounded and that the ellipticity condition

$$\mu \sum_{i=1}^{m} \xi_i^2 \leqslant \sum_{i,k=1}^{m} a_{ik}(x)\, \xi_i \xi_k \leqslant M \sum_{i=1}^{m} \xi_i^2 \text{ for every } x \text{ and } \xi,$$
$$0 < \mu = \text{const} \leqslant M = \text{const}. \tag{2.11}$$

holds. Let P be a measurable function defined in $R^m \times R^m$, such that

(i) $P(x, \xi) \geqslant 0$ for all x and ξ.

(ii) $\int_{R^m} P(x, \xi)\, d\xi = 1$ for all x.

(iii) The first moments vanish, i.e., $\int_{R^m} P(x, \xi)\, \xi_k\, d\xi = 0$ for $k = 1, 2,..., m$ and for every x. This condition is automatically verified if P is an even function of ξ.

(iv) A constant C exists such that $\int_{R^m} P(x, \xi)\, \xi_i \xi_k\, d\xi = 2Ca_{ik}(x)$ $(i, k = 1, 2,..., m)$.

(v) $\int_{R^m} (1 + |\xi|^2)[\text{ess sup}_{x \in R^m} P(x, \xi)]\, d\xi < +\infty$, a condition that guarantees the uniform convergence of the integrals appearing in (ii)–(iv).

In Section 2.3, we give several examples of kernels verifying (i)–(v).

Let $0 \leqslant t \to A(t)$ be the operator valued function defined by $A(0) =$ the identity operator, and if $t > 0$,

$$A(t)\, u(x) = t^{-m/2} \int_{R^m} P(x, (x - y)/t^{1/2})\, u(y)\, dy. \tag{2.12}$$

From (i) and (ii) we have

$$\| A(t)u \|_{L^\infty(R^m)} \leqslant \| u \|_{L^\infty(R^m)} = \text{ess sup}\, | u |, \tag{2.13}$$

so that $A(t)$ is a contraction on $L^\infty(R^m)$.

On the other hand, it is easy to see that $A(t)$ is a bounded operator in $L^1(R^m)$. In fact,

$$\| A(t)u \|_{L^1(R^m)} \leqslant n(t)\, \| u \|_{L^1(R^m)}, \tag{2.14}$$

where

$$n(t) = \operatorname*{ess\,sup}_{y \in R^m} t^{-m/2} \int_{R^m} P(x, (x - y)/t^{1/2})\, dx, \tag{2.15a}$$

or what amounts to the same thing:

$$n(t) = \operatorname*{ess\,sup}_{y \in R^m} \int_{R^m} P(y + t^{1/2}\xi, \xi)\, d\xi. \tag{2.15b}$$

Clearly,

$$n(t) \leqslant \int_{R^m} [\operatorname*{ess\,sup}_{x \in R^m} P(x, \xi)]\, d\xi \qquad \text{for all} \quad t > 0,$$

this estimate being a finite constant by virtue of (v).

By interpolation (or by an appropriate use of Hölder's inequality), $A(t)$ is a bounded operator in $L^p(R^m)$ $(1 \leqslant p \leqslant +\infty)$ and

$$\| A(t)u \|_{L^p(R^m)} \leqslant n(t)^{1/p} \| u \|_{L^p(R^m)}, \tag{2.16}$$

where $n(t)$ is given by (2.15).

Consider the derivative at $t = 0$ of the operator-valued function $0 \leqslant t \to A(t)$. Here, we think of $A(t)$ as a bounded operator acting in $L^p(R^m)$. We denote by $C \cdot E$ this derivative, C being the constant appearing in (iv); thus, E is defined in the following way.

(i) The domain $D(E)$ is the collection of all functions u in $L^p(R^m)$ such that the differential quotient $(A(t)u - u)/t$ has a limit in $L^p(R^m)$ as $t \downarrow 0$.

(ii) For every u in $D(E)$, Eu is the $L^p(R^m)$-limit of $(A(t)u - u)/Ct$ for $t \downarrow 0$.

THEOREM 2.1. *The operator E, defined by* (i) *and* (ii) *above, is an extension of the differential operator*

$$\sum_{i,k=1}^{m} a_{ik}\partial^2/\partial x_i\, \partial x_k : L^p(R^m) \supset W^{2,p}(R^m) \to L^p(R^m). \tag{2.18}$$

In other words, $D(E) \supseteq W^{2,p}(R^m)$ and for every $u \in W^{2,p}(R^m)$ Eu has the expression $Eu = \sum_{i,k=1}^{m} a_{ik}\, \partial^2 u/\partial x_i\, \partial x_k$.

The theorem holds for $1 \leqslant p \leqslant +\infty$. For us, $W^{2,\infty}(R^m)$ is the set of all bounded functions in $C^2(R^m)$ whose second derivatives are bounded and uniformly continuous.

The proof is quite simple. Consider, for instance, the case $1 \leqslant p < +\infty$ and let u be any function in $W^{2,p}(R^m)$.

From (2.12) we get, with an obvious change of variables,

$$A(t)\, u(x) = \int_{R^m} P(x, \xi)\, u(x - t^{1/2}\xi)\, d\xi.$$

On the other hand, Taylor's formula reads

$$u(x + y) = u(x) + (Du(x), y) + \tfrac{1}{2}(D^2 u(x)y, y) + R(x, y)| y |^2,$$

$$\int_{R^m} | R(x, y)|^p\, dx \begin{cases} \text{bounded in } y, \\ \to 0 \text{ if } | y | \to 0. \end{cases}$$

We denote Du the gradient and D^2u the Hessian matrix of u, so that

$$(Du(x), y) = \sum_{i=1}^{m} u_{x_i}(x)\, y_i\,, \qquad (D^2u(x)y, y) = \sum_{i,k=1}^{m} u_{x_i x_k}(x)\, y_i y_k\,.$$

Hence, we obtain

$$A(t)\, u(x) = u(x) \int_{R^m} P(x, \xi)\, d\xi - t^{1/2} \sum_{k=1}^{m} u_{x_k}(x) \int_{R^m} P(x, \xi)\, \xi_k\, d\xi$$

$$+ (t/2) \sum_{i,k=1}^{m} u_{x_i x_k}(x) \int_{R^m} P(x, \xi)\, \xi_i \xi_k\, d\xi + (\text{a remainder}).$$

That is, recalling hypotheses (ii)–(iv) on the kernel P:

$$A(t)\, u(x) - u(x) = Ct \sum_{i,k=1}^{m} a_{ik}(x)\, u_{x_i x_k}(x) + (\text{a remainder})$$

The remainder term is

$$t \int_{R^m} P(x, \xi) \mid \xi \mid^2 R(x, t^{1/2}\xi)\, d\xi,$$

which can be estimated using the Hölder inequality by

$$t \left(\int_{R^m} P(x, \xi) \mid \xi \mid^2 d\xi \right)^{1-(1/p)} \left(\int_{R^m} P(x, \xi) \mid \xi \mid^2 \mid R(x, t^{1/2}\xi) \mid^p d\xi \right)^{1/p}$$

$$\leqslant t \left(\int_{R^m} \operatorname*{ess\,sup}_{y \in R^m} P(y, \xi) \mid \xi \mid^2 d\xi \right)^{1-(1/p)}$$

$$\times \left(\int_{R^m} \operatorname*{ess\,sup}_{y \in R^m} P(y, \xi) \mid \xi \mid^2 \mid R(x, t^{1/2}\xi) \mid^p d\xi \right)^{1/p}$$

$$= t \left(\int_{R^m} \mid R(x, t^{1/2}\xi) \mid^p \times (\text{some integrable function } N(\xi))\, d\xi \right)^{1/p}$$

(cf. condition (v) on P). Thus,

$$\| \text{the remainder} \|_{L^p(R^m)} \leqslant t \left(\int_{R^m} \| R(\cdot, t^{1/2}\xi) \|_{L^p(R^m)}\, N(\xi)\, d\xi \right)^{1/p}$$

$$= o(t), \qquad \text{if} \quad t \downarrow 0,$$

by the dominated convergence theorem. The proof is complete.

Theorem 2.1 enables us to define a kind of generalized solution for elliptic equations. The question arises: Under what smoothness hypotheses on the coefficients will these generalized solutions actually be (in some sense) strong solutions? The following theorem gives a simple answer.

THEOREM 2.2. *Suppose the coefficients $a_{ik}(x)$ are twice continuously differentiable with bounded uniformly continuous second derivatives, the ellipticity condition* (2.11) *holding. Let us define the operator E as before; here the kernel P has to be choosen* (as can be done; see Section 2.3) *in such a way that*

(i) $P(x, \xi) = 0$ *for all* x *if* $|\xi|$ *is large enough,*

(ii) *the second derivatives of P with respect to x are uniformly continuous in* $R^m \times R^m$.

Then, $D(E)$(the domain of E) $= W^{2,p}(R^m)$; *hence, E coincides with the differential operator* (2.18). *Here,* $1 < p < +\infty$.

A proof of this theorem is based on the following lemmas, the first of which is well known (see [44, par. 37]).

LEMMA 2.3. *Suppose that the coefficients $a_{ik}(x)$ are uniformly continuous and that the ellipticity condition* (2.11) *holds. Then, if* $1 < p < +\infty$:

$$\| u \|_{W^{2,p}(R^m)} \leqslant (\text{const independent of } u)$$
$$\times \left(\left\| \sum_{i,k=1}^{m} a_{ik} u_{x_i x_k} \right\|_{L^p(R^m)} + \| u \|_{L^p(R^m)} \right),$$

for every "test" function u.

For a quick proof of this lemma see [11].

LEMMA 2.4. *Suppose that the matrix $(a_{ik}(x))$ has nonnegative eigenvalues, that $a_{ik} \in C^2(R^m)$, and that*

$$\sum_{i,k=1}^{m} \partial^2 a_{ik}(x)/\partial x_i \, \partial x_k \leqslant B = \text{const}.$$

Then, if $1 < p < +\infty$ *and* λ *is any number greater than* B/p, *the inequalities*

$$\| u \|_{L^p(R^m)} \leqslant \frac{1}{\lambda - (B/p)} \left\| \sum_{i,k=1}^{m} a_{ik} u_{x_i x_k} - \lambda u \right\|_{L^p(R^m)}$$

and

$$\| u \|_{L^{p'}(R^m)} \leqslant \frac{1}{\lambda - (B/p)} \left\| \sum_{i,k=1}^{m} \frac{\partial^2}{\partial x_i \, \partial x_k} (a_{ik} u) - \lambda u \right\|_{L^{p'}(R^m)}$$

hold for every "test" function u. Here, $p' = p/(p-1)$.

The first inequality of Lemma 2.4 is easily proved in the case $p \geqslant 2$ by integrations by parts, the second can be proved by similar devices in the case $1 < p \leqslant 2$; the proof in the other cases requires a duality argument. For the sake of brevity we omit the details.

LEMMA 2.5. *The hypotheses on the* $a_{ik}(x)$ *and on the kernel P are as in Theorem* 2.2. *Then, the function* $n(t)$ *defined in* (2.15) *is differentiable at* $t = 0$, $n(0) = 1$, *and*

$$(1/C)\, n'(0) = B = \sup_{x \in R^m} \sum_{i,k=1}^{m} \partial^2 a_{ik}(x)/\partial x_i \, \partial x_k \, . \tag{2.19}$$

Later, we give a proof of Lemma 2.5. Here we sketch a proof of Theorem 2.2.

The key of the argument is the following. If we shift E into $E - (B/p)$, where B is defined by (2.19), then $C(E - (B/p))$ is the derivative at $t = 0$ of an operator-valued function whose values are *contractions* on $L^p(R^m)$. Such an operator-valued function is $0 \leqslant t \to n(t)^{-1/p} A(t)$, where $n(t)$ is given by (2.15). To see this, we have only to remember the definition of E, Lemma 2.5 and to look at (2.16). Consequently, we can claim: the operator $E - \lambda$ (from $D(E)$ into $L^p(R^m)$) is one-to-one for every $\lambda > B/p$. See Section 1.1 for a proof.

Now, let u be any function in $D(E)$ and let v be the (unique) solution of

$$v \in W^{2,p}(R^m), \qquad \sum_{i,k=1}^{m} a_{ik} v_{x_i x_k} - \lambda v = Eu - \lambda u,$$

where λ is any fixed constant $> B/p$. The solution v exists by virtue of Lemmas 2.3 and 2.4.

Put $u = v + w$. Then, from Theorem 2.1, we infer

$$w \in D(E), \qquad Ew - \lambda w = 0.$$

As $E - \lambda$ is one-to-one, w must be zero, hence, $u = v \in W^{2,p}(R^m)$.

Thus, we have proved $D(E) \subseteq W^{2,p}(R^m)$. The conclusion follows from Theorem 2.1.

Proof of Lemma 2.5. It is apparent from (2.15b) that $n(0) = 1$. Taylor's expansion of the function $\lambda \to \int_{R^m} P(x + \lambda\xi, \xi)\, d\xi$ gives

$$\int_{R^m} P(x + t^{1/2}\xi, \xi)\, d\xi = \int_{R^m} P(x, \xi)\, d\xi + t^{1/2} \sum_{k=1}^{m} \int_{R^m} \frac{\partial P}{\partial x_k}(x, \xi)\, \xi_k\, d\xi$$

$$+ \frac{t}{2} \sum_{i,k=1}^{m} \int_{R^m} \frac{\partial^2 P}{\partial x_i\, \partial x_k}(x, \xi)\, \xi_i \xi_k\, d\xi + (\text{a remainder})$$

and the hypotheses made on P yield

$$\int_{R^m} \frac{\partial P}{\partial x_k}(x, \xi)\, \xi_k\, d\xi = 0, \qquad \int_{R^m} \frac{\partial^2 P}{\partial x_i\, \partial x_k}(x, \xi)\, \xi_i \xi_k\, d\xi = 2C\, \frac{\partial^2 a_{ik}}{\partial x_i\, \partial x_k}(x),$$

the remainder $= o(t)$ for $t \downarrow 0$ *uniformly* in x. Hence,

$$\frac{1}{t}\left(\int_{R^m} P(x + t^{1/2}\xi, \xi)\, d\xi - 1\right) - C \sum_{i,k=1}^{m} \frac{\partial^2 a_{ik}}{\partial x_i\, \partial x_k}(x) \to 0,$$

for $t \downarrow 0$ *uniformly* in x. Consequently,

$$\frac{1}{t}(n(t) - 1) - C \sup_{x \in R^m} \sum_{i,k=1}^{m} \frac{\partial^2 a_{ik}}{\partial x_i\, \partial x_k}(x) \to 0 \qquad \text{for} \quad t \downarrow 0.$$

2.3. Here, we give some examples of kernels verifying conditions (i)–(v) listed at the beginning of the previous subsection.

In what follows, φ indicates any real-valued function of a scalar variable $r \geqslant 0$, such that

$$\varphi(r) \geqslant 0,$$

$$\int_0^{+\infty} \varphi(r)\, r^{m-1}\, dr = 1/\omega_m\,, \tag{2.25}$$

$$(\omega_m/2m) \int_0^{+\infty} r^2 \varphi(r)\, r^{m-1}\, dr = C < \infty.$$

EXAMPLE 1. This example is suggested by generalizations of formula (2.7).

$$P(x, \xi) = \varphi(|\, a(x)^{-1/2}\, \xi\, |)(\det a(x))^{-1/2}. \tag{2.26}$$

In this formula (and in the following ones), $a(x)$ is the (symmetric) matrix $(a_{ik}(x))_{i,k=1,\ldots,m}$; $a(x)^{-1/2}$ is the positive square root of $a(x)^{-1}$ (= the inverse of $a(x)$); $a(x)^{-1/2}\xi$ is the action of $a(x)^{-1/2}$ on the vector ξ; and $|\, a(x)^{-1/2}\xi\, | = ((a(x)^{-1}\xi, \xi))^{1/2}$ is its length.

Properties (i)–(iii) of P are quite obvious. To prove (iv), we note that

$$\begin{aligned}\int_{R^m} \varphi(|\, \xi\, |)\, \xi_i \xi_k\, d\xi &= \int_0^{+\infty} r^2 \varphi(r)\, r^{m-1}\, dr \int_{|\xi|=1} \xi_i \xi_k H_{m-1}(d\xi) \\ &= (\omega_m/m)\, \delta_{ik} \int_0^{+\infty} r^2 \varphi(r)\, r^{m-1}\, dr = 2C\delta_{ik}\,.\end{aligned}$$

Hence, if U is any $m \times m$ matrix:

$$\begin{aligned}&\int_{R^m} \varphi(|\, a(x)^{-1/2}\, \xi\, |)(\det a(x))^{-1/2}\, (U\xi, \xi)\, d\xi \\ &\quad = \int_{R^m} \varphi(|\, \xi\, |)([a(x)^{1/2}\, U a(x)^{1/2}]\xi, \xi)\, d\xi \\ &\quad = 2C \cdot \text{trace of } a(x)^{1/2}\, U a(x)^{1/2} \\ &\quad = 2C \cdot \text{trace of } a(x)^{-1/2}\, [a(x)U]\, a(x)^{1/2} \\ &\quad = 2C \cdot \text{trace of } a(x)U.\end{aligned}$$

To verify condition (v), we add to (2.25) a slightly more stringent hypothesis on φ. It is easy to see that (v) holds if

$$\int_0^{+\infty} (1 + r^2) \sup\{\varphi(s) : r \leqslant s \leqslant kr\}\, r^{m-1}\, dr < \infty, \tag{2.27}$$

where $k = M/\mu$ is the ratio between the ellipticity constants (2.11).

Note that (2.27) follows from (2.25) if φ is decreasing; the same is true if φ has bounded variation and

$$\int_0 r\, |\, \varphi'(r)|\, r^{m-1}\, dr < \infty, \qquad \int^{+\infty} r^3\, |\, \varphi'(r)|\, r^{m-1}\, dr < \infty.$$

EXAMPLE 2.

$$P(x, \xi) = \frac{\pi^{1/2}\,\Gamma(n+1+(m/2))}{\Gamma(1+(m/2))\,\Gamma(n+1/2)\,2n}\,\varphi(|\xi|)$$
$$\times |\xi|^{-2n} \sum_{i=1}^{m} \Big(\lambda_i(x) - \frac{m/2}{n+(m/2)}\Big)\Big(\sum_{k=1}^{m} U_{ik}(x)\,\xi_k\Big)^{2n}. \tag{2.28}$$

Here, $0 < \lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_m(x)$ are the eigenvalues of the matrix $a(x) = (a_{ik}(x))_{i,k=1,\ldots,m}$ and $U(x) = (U_{ik}(x))_{i,k=1,\ldots,m}$ is an *orthogonal* matrix such that

$$U(x)\,a(x)\,U(x)^* = \begin{pmatrix} \lambda_1(x) & & & 0 \\ & \lambda_2(x) & & \\ & & \ddots & \\ 0 & & & \lambda_m(x) \end{pmatrix}, \tag{2.29}$$

(where $*$ = transposed). We assume that the normalization condition

$$\sum_{i=1}^{m} a_{ii}(x) \equiv \sum_{i=1}^{m} \lambda_i(x) = m.$$

holds. The number n is any positive integer; we can freely choose n large enough so that $P \geqslant 0$, i.e.,

$$(m/2)/(n+(m/2)) < \text{the ellipticity constant } \mu.$$

In other words, we see that for a kernel of the form (2.28) to be positive, the degree n of the polynomial in the direction cosines $\xi_k/|\xi|$ must be sufficiently large, depending on the smallest ellipticity constant. If n is fixed in advance, there exist elliptic differential operators for which kernels of the form (2.28) do not have constant sign. This situation has an analog in [45]. These authors consider the problem of approximating a given second-order linear elliptic differential operator by means of difference operators with positive coefficients. They prove that the difference scheme must be chosen depending on the ellipticity of the differential operator; i.e., a difference scheme that works for every elliptic differential operator does not exist.

Properties (iii) and (v) of the kernel are obvious; (ii) and (iv) follow from the formulas

$$\int_{R^m} P(x, \xi)\,d\xi = \int_{R^m} P(x, U(x)^*\,\xi)\,d\xi,$$

the matrix

$$\left(\int_{R^m} P(x, \xi)\, \xi_i \xi_k \, d\xi\right)_{i,k=1,\ldots,m}$$

$$= U(x)^* \left(\int_{R^m} P(x, U(x)^* \xi)\, \xi_i \xi_k \, d\xi\right)_{i,k=1,\ldots,m} U(x);$$

$$P(x, U(x)^* \xi) = \frac{\pi^{1/2}\, \Gamma(n + 1 + (m/2))}{\Gamma(1 + (m/2))\, \Gamma(n + \frac{1}{2})\, 2n}\, \varphi(|\xi|)$$

$$\times \sum_{i=1}^{m} \left(\lambda_i(x) - \frac{m/2}{n + (m/2)}\right) (\xi_i / |\xi|)^{2n},$$

$$\int_{|\xi|=1} \xi_i^{2n} H_{m-1}(d\xi) = \frac{2\pi^{(m-1)/2}\, \Gamma(n + \frac{1}{2})}{\Gamma(n + (m/2))},$$

$$\int_{|\xi|=1} \xi_i^{2n} \xi_j \xi_k H_{m-1}(d\xi) = \frac{\pi^{(m-1)/2}\, \Gamma(n + \frac{1}{2})}{\Gamma(n + 1 + (m/2))} (2n\delta_{ij}\delta_{ik} + \delta_{jk}).$$

The last two formulas imply

$$\int_{R^m} P(x, U(x)^* \xi)\, d\xi = 1,$$

$$\begin{aligned} \int_{R^m} P(x, U(x)^* \xi)\, \xi_i \xi_k \, d\xi &= 0 && \text{if } i \neq k, \\ &= 2C\lambda_k(x) && \text{if } i = k. \end{aligned}$$

EXAMPLE 3.

$$P(x, \xi) = -\omega_m \varphi(|\xi|) \sum_{i,k=1}^{m} a_{ik}(x) \frac{\xi_i}{|\xi|} \frac{\partial G}{\partial y_k}\left(x, \frac{\xi}{|\xi|}\right), \tag{2.35}$$

where G is the map from $R^m \times \{y \in R^m: 0 < |y| \leqslant 1\}$ into the (nonnegative) reals defined by the following rule. For every x, $G(x, \cdot)$ is the infinitely differentiable solution of the (constant coefficient) Dirichlet problem:

$$\begin{aligned} \sum_{i,k=1}^{m} a_{ik}(x) \frac{\partial^2 G}{\partial y_i\, \partial y_k}(x, y) &= \text{the Dirac mass at } y = 0, \\ G(x, y) &= 0 \quad \text{if } |y| = 1. \end{aligned} \tag{2.36}$$

The Dirac mass in (2.36) is the effect of the constant coefficient operator $\sum a_{ik}(x)\, \partial^2/\partial y_i\, \partial y_k$ on its fundamental solution:

$$(1/(m-2)\,\omega_m)(\det a(x))^{-1/2}\,(a(x)^{-1}y, y)^{1-(m/2)} \qquad (\text{if } m \geqslant 3),$$
$$(1/2\pi)(\det a(x))^{-1/2}\ln(a(x)^{-1}y, y)^{-1/2} \qquad (\text{if } m = 2).$$

In other words, $G(x, y) =$ (the fundamental solution) $+\, v(x, y)$, where $v(x, \cdot)$ is a smooth solution of:

$$\sum a_{ik}(x)(\partial^2 v/\partial y_i\, \partial y_k)(x, y) = 0.$$

Or, $G(x, y) = K(x; y, 0)$, where $K(x; y, z)$ is the Green's function in the unit ball for the aforesaid operator. Note that the right-hand side of (2.35), apart from the factor $-\omega_m\varphi(|\,\xi\,|)$, is the conormal derivative (at the boundary point $\xi/|\,\xi\,|$) of the solution of (2.36).

As in the previous example, we normalize the coefficients in the following way:

$$\sum_{i=1}^{m} a_{ii}(x) = m. \tag{2.38}$$

Property (v) of P is easy to check. Properties (i)–(iv) follow from Stoke's theorem, which gives

$$\begin{aligned} u(0) = &- \int_{|y|=1} u(y) \sum_{i,k=1}^{m} a_{ik}(x)\, y_i\, \frac{\partial G}{\partial y_k}(x, y)\, H_{m-1}(dy) \\ &- \int_{|y|<} \sum_{i,k=1}^{m} a_{ik}(x)\, \frac{\partial^2 u}{\partial y_i\, \partial y_k}(y)\, G(x, y)\, dy, \end{aligned} \tag{2.39}$$

where u is any smooth function. Then, (2.39), coupled with the maximum principle, easily shows (i), that is, the positiveness of P. In (2.39), replace $u(y)$ by $u(ry)$, multiply both members by $\omega_m\varphi(r)r^{m-1}$, and then integrate over r. One obtains

$$\begin{aligned} u(0) = &\int_{R^m} P(x, \xi)\, u(\xi)\, d\xi \\ &- \omega_m \int_0^{+\infty} r\varphi(r)\, dr \int_{|y|<r} \sum_{i,k=1}^{m} a_{ik}(x)\, \frac{\partial^2 u}{\partial y_i\, \partial y_k}(y)\, G\left(x, \frac{y}{r}\right) dy. \end{aligned} \tag{2.40}$$

Clearly, substituting $u(y) = 1$ or $u(y) = y_i$ into (2.39), we obtain properties (ii) and (iii). Substituting $u(y) = y_i y_k$, into (2.40), we see that

$$\int_{R^m} P(x, \xi)\, \xi_i \xi_k \, d\xi = 2a_{ik}(x)\, \omega_m \int_0^{+\infty} r^2 \varphi(r)\, r^{m-1}\, dr \int_{|y|<1} G(x, y)\, dy$$

$$= 2Ca_{ik}(x)\, 2m \int_{|y|<1} G(x, y)\, dy,$$

On the other hand, the choice $u(y) = 1 - |y|^2$ in (2.39) gives

$$\int_{|y|<1} G(x, y)\, dy = 1\Big/2\sum_{i=1}^{m} a_{ii}(x) = 1/2m,$$

by the normalization (2.38). Thus, property (iv) also is proved.

As far as an explicit representation formula for G is concerned, we can make the following remarks. Simple changes of coordinates show that G can be computed by means of the Green's function for the Laplace operator in an ellipsoid, namely,

$$G(x, y) = K(U(x)\, a(x)^{-1/2}\, y;\, (\lambda_1(x))^{-1/2}, \ldots, (\lambda_m(x))^{-1/2}),$$

where $U(x)$ is an orthogonal matrix such that $U(x)\, a(x)\, U(x)^*$ is diagonal and the $\lambda_k(x)$ are the eigenvalues of $a(x)$. $K(x; r_1, \ldots, r_m)$ is the solution u of

$$\sum_{k=1}^{m} (\partial^2 u/\partial x_k^2) = \text{the Dirac mass at the origin} \left(\text{in the ellipsoid } \sum_{k=1}^{m} x_k^2/r_k^2 < 1\right),$$

$$u = 0, \qquad \left(\text{on the boundary } \sum_{k=1}^{m} x_k^2/r_k^2 = 1\right).$$

In two dimensions ($m = 2$), the expression of K can be deduced easily from Ghizzetti's result [24]. The formula is

$$K(x_1, x_2; r_1, r_2) = -\frac{1}{2\pi} \ln r - \frac{1}{2\pi} \sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \frac{r^{2n} - r^{-2n}}{q^{2n} + q^{-2n}} \cos(2n\vartheta),$$

where $q = |(r_1 - r_2)/(r_1 + r_2)|^{1/2}$ and $re^{i\vartheta}$ is related to $x = x_1 + ix_2$ by the conformal map: $2x = (r_1 + r_2)\, re^{i\vartheta} + (r_1 - r_2)\, e^{-i\vartheta}/r$.

EXAMPLE 4. The kernels of the previous examples are of the form: (A radial function depending on the length $|\xi|$ only) $\times$ (a function depending on x and on the direction $\xi/|\xi|$ only). All kernels of this form can be written as $\varphi(|\xi|) \times$ (an expansion in spherical harmonics evaluated at $\xi/|\xi|$). It is easy to see that the beginning of such expansion is determined uniquely by conditions (ii)–(iv), while the remainder has the sole task of assuring positiveness.

To be precise, the example we have in mind is:

$$P(x, \xi) = \varphi(|\xi|) + ((m+2)/2)\, \varphi(|\xi|) \sum_{i,k=1}^{m} (a_{ik}(x) - \delta_{ik})(\xi_i \xi_k / |\xi|^2)$$

$$+ \sum_{k=3}^{+\infty} \sum_{j=1}^{N(k,m)} b_{kj}(x)\, \varphi(|\xi|)\, Y_{kj}(\xi/|\xi|), \tag{2.45}$$

where φ is as before. We suppose again that the coefficients are normalized in the following way:

$$\sum_{i=1}^{m} a_{ii}(x) = m. \tag{2.46}$$

The sequence $\{Y_{kj}\}_{j=1,\ldots,N(k,m)}$ is a complete orthonormal set of spherical harmonics of degree k; $N(k, m)$ is the number of linearly independent spherical harmonics of degree k in m dimensions. We emphasize that the series in (2.45) starts with $k = 3$. Note that by virtue of (2.46) the second term in the right-hand side of (2.45),

$$\sum_{i,k=1}^{m} (a_{ik}(x) - \delta_{ik})\, \xi_i \xi_k\,,$$

is a harmonic quadratic polynomial in ξ.

The b_{kj} are real-valued measurable functions such that

$$\sum_{k=3}^{+\infty} \left(N(k, m) \sum_{j=1}^{N(k,m)} \operatorname*{ess\,sup}_{x \in R^m} b_{kj}(x)^2 \right)^{1/2} < \infty. \tag{2.47}$$

According to the well-known formula

$$\omega_m \sum_{j=1}^{N(k,m)} Y_{kj}(\xi/|\xi|)^2 = N(k, m),$$

(2.47) guarantees property (v). Note that we can choose $b_{kj} = 0$ if and only if the beginning of the right-hand side of (2.45) is positive, a case that occurs if and only if the ellipticity constant $\mu > m/(m+2)$.

Properties (ii)–(iv) follow at once from the orthogonality between spherical harmonics of different degree; from the equation

$$\xi_i \xi_k = [\xi_i \xi_k - (\delta_{ik}/m)(\xi_1^2 + \cdots + \xi_m^2)] + (\delta_{ik}/m)$$

$$= \text{a harmonic polynomial} + \text{a constant}$$

$$\text{on the unit sphere } \xi_1^2 + \cdots + \xi_m^2 = 1,$$

and from the formulas

$$\int_{|\xi|=1} \xi_i \xi_k H_{m-1}(d\xi) = \omega_m \delta_{ik}/m,$$

$$\int_{|\xi|=1} \xi_i \xi_j \xi_h \xi_k H_{m-1}(d\xi) = (\omega_m/m(m+2))(\delta_{ij}\delta_{hk} + \delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}).$$

Note that if P has the form (2.31), the operator $A(t)$ (2.12) can be split into the sum

$$A(t) = [\text{the convolution } t^{-m/2}\varphi(t^{-1/2}\,|\cdot|)*] + B(t) + R(t),$$

where the first term is the operator associated with the Laplacian (see Section 1.2) and

$$B(t)\,u(x) = \frac{m+2}{2}\, t^{-m/2} \sum_{i,k=1}^{m} (a_{ik}(x) - \delta_{ik})$$

$$\times \int_{R^m} \varphi(t^{-1/2}\,|\,x-y\,|)\, \frac{(x_i - y_i)(x_k - y_k)}{|\,x-y\,|^2}\, u(y)\, dy.$$

Consequently, if u is a function in some $W^{2,p}(R^m)$, we can write

$$(1/Ct)(A(t)u - u) = (1/Ct)(t^{-m/2}\varphi(t^{-1/2}\,|\cdot|) * u - u) + (1/Ct)\, B(t)u$$

and Theorem 2.1 shows that, as $t \downarrow 0$,

(i) the first term tends to Δu in $L^p(R^m)$,

(ii) $(1/Ct)\, B(t)u \to \sum_{i,k=1} (a_{ik} - \delta_{ik})\, u_{x_i x_k}$,

(iii) $(1/Ct)\, R(t)u$, the remainder containing the "arbitrary part" of P, tends to 0 in $L^p(R^m)$.

3. On the Maximum Principle

3.1. The maximum principle (in one of its forms) states that no real-valued nonconstant solution u of an uniformly elliptic equation of the form $Eu \equiv \sum_{i,k=1}^{m} a_{ik}(x)\, u_{x_i x_k} = 0$ has local extrema.

This fact was proved a long time ago by E. Hopf (see [44]) in the case of twice continuously differentiable solutions.

The maximum principle for solutions belonging to Sobolev spaces $W^{2,p}$ was proved by Aleksandrov [1] and in [49, 50] in the case $p = m$ (= number of dimensions) and by Bony [8] in the case $p > m$.

In this section, we prove the maximum principle for a class of solutions, related to the generalized solutions introduced in the previous section.

The solutions we have in mind are in the class of the continuous functions u, defined on open sets $G \subseteq R^m$, such that $Eu \in L^1_{\mathrm{loc}}(G)$. In addition, they have to verify the following:

(i) Assume u is differentiable (in the usual sense) at every point of G. Moreover, Du (the gradient of u, a map of G into R^m) has the following property, somewhat related to absolute continuity: Du maps subsets of G of Lebesgue measure zero into subsets of R^m of measure zero.

For a sufficiently smooth (locally integrable for instance, or even continuous) function u on an open set G, we write $Eu \in L^1_{\mathrm{loc}}(G)$ to mean: Eu is actually a function in $L^1_{\mathrm{loc}}(G)$, which is related to u by a special procedure. The latter point is essential, of course. More precisely, we mean: u can be related to some function in $L^1_{\mathrm{loc}}(G)$, which we call Eu, by the rule

$$\int_K | Eu(x) - (1/Ct)(A(t)\, u(x) - u(x))|\, dx \to 0 \qquad (t \downarrow 0)$$

for every compact set $K \subset G$.

Here, the average $A(t)$ and its kernel $P(x, \xi)$ are defined as in Section 2.1 and in addition,

$$\text{support of } P(x,\ ) \subseteq \text{some compact set independent of } x, \tag{3.2}$$

a condition guaranteeing that for every compact set $K \subset G$, the average $A(t)\, u(x)$ is well defined at almost every point $x \in K$ for any sufficiently small t.

The condition on Du stated in (i) is reported by Radò and Reichelderfer [55, Section IV. 1.4] as condition (N), in connection with continuous mappings of R^m into itself. Note that (i) is certainly true if u is in $W^{2,p}(G)$ for some $p > m$. For, $u \in W^{2,p}(G)$ and $p > m$ imply that u is continuously differentiable (perhaps after a correction on a set of measure zero; as a matter of a fact u has Hölder continuous first derivatives). On the other hand, $\det D^2u(x)$, the Hessian determinant (= the Jacobian of Du) is absolutely integrable on sets $M \subseteq G$ of finite measure and the formula

$$\int_M |\det D^2u(x)|\, dx = \int_{Du(M)} \operatorname{card}\{x \in M : Du(x) = y\}\, dy \geqslant \operatorname{meas} Du(M).$$

holds. For another proof, see [8].

Assumption (i) can be relaxed. Indeed, we can assume in place of (i) the following.

(ii) For every set $M \subset G$ such that $\operatorname{meas} M = 0$, the sets

$$\bigcup_{x \in M} \{h \in R^m : (h, y - x) + u(x) \geqslant u(y) \text{ for every } y \in G\}$$

and

$$\bigcup_{x \in M} \{h \in R^m : (h, y - x) + u(x) \leqslant u(y) \text{ for every } y \in G\}$$

have measure zero.

It is quite clear that (i) implies (ii). Aleksandrov [1, V] proved that (ii) is true if $u \in W^{2,m}(G)$.

THEOREM 3.1. *Let u be a real-valued continuous function, which is a solution of a homogeneous uniformly elliptic equation $Eu = 0$, with leading terms only and bounded measurable coefficients. Here, Eu is to be taken in the sense* (3.1) *explained before. Suppose further, that u has one of the previous properties* (i) *or* (ii). *Then, u has no local strict extrema, unless u is constant.*

We sketch the idea of the proof. To avoid technical difficulties, let us refer to the simpler hypothesis (i), since the proof under the more sophisticated hypothesis (ii) goes almost along the same lines.

Suppose, if possible, that u reaches a strict maximum at a point $a \in G$. Then, the function $v(x) = u(x) + \lambda\, |\, x - a\, |^2$ has a strict maximum at some point $b \in G$ near to a, if λ is positive and small enough. A direct computation (remember (iv) in Section 2.2) shows that $Ev = Eu + 2\lambda \sum_{i=1}^{m} a_{ii} = 2\lambda \sum_{i=1}^{m} a_{ii} \geqslant$ constant > 0 a.e.

Henceforth, the situation is the following. We have a function v, verifying (i), such that Ev is a strictly positive $L^1_{loc}(G)$-function. Moreover, v attains a strict maximum at a point $b \in G$.

Let M be the set of all points of concavity for v, i.e.,

$$M = \{x \in G : v(x) + (Dv(x), y - x) \geqslant v(y) \text{ for every } y \text{ in a neighborhood of } x\}.$$

The key to the proof is the following.

Claim. meas $M > 0$.

We shall prove this later by showing that the image $Dv(M)$ has positive measure. Indeed, in consequence of hypothesis (i), we cannot have meas $M = 0$ if meas $Dv(M) > 0$.

Assuming the claim, we immediately achieve a contradiction. In fact, if $x \in M$, keeping in mind (3.2), we have for every sufficiently small t:

$$A(t)\, v(x) - v(x)$$
$$\leqslant t^{-m/2} \int_G P(x, (x-y)/t^{1/2})[v(x) + (Dv(x), y - x)]\, dy - v(x) = 0,$$

the equality being a consequence of properties (ii) and (iii) (see Section 2.2) of P. Then, taking account of the definition of Ev (cf. (3.1)) we must have $Ev(x) \leqslant 0$ for almost all $x \in M$. This is impossible, since we know that Ev is a.e. strictly positive and meas $M > 0$.

Now, to prove that meas $Dv(M) > 0$, we show that $Dv(M)$ contains a ball.

Let B be an open ball with center at b ($=$ the extremal point of v) and radius $r <$ distance$(b, \partial G)$ such that

$$v(b) > \max_{x \in \partial B} v(x).$$

We state that

$$Dv(M) \supseteq \{h \in R^m : |h| < (1/r)(v(b) - \max_{x \in \partial B} v(x))\}. \tag{3.3}$$

In fact, consider the function

$$w(x) = v(x) - (h, x - b).$$

Clearly,

$$\max_{x\in\partial B} w(x) - w(b) \leqslant \max_{x\in\partial B} v(x) - v(b) + r \mid h \mid,$$

so w attains its maximum value in $\bar{B}$ at an interior point, provided

$$\mid h \mid < (1/r)(v(b) - \max_{x\in\partial B} v(x)). \tag{3.4}$$

Let x be such a point; from the definition of w, we get

$$v(x) + (h, y - x) \geqslant v(y) \text{ for every } y \in B \text{ } (= \text{a neighborhood of } x).$$

As v is differentiable at the point x, this inequality implies $Dv(x) = h$; hence, $x \in M$.

Thus, we have proved that for every $h \in R^m$, verifying (3.4), there exists $x \in M$ such that $Dv(x) = h$; so (3.3) is true. The proof is complete.

Remark. We mention here a version of the maximum principle for functions that are solutions in a generalized *pointwise* sense of linear homogeneous elliptic equations.

Let u be a real-valued continuous function on an open connected set $G \subseteq R^m$. Suppose that at every point $x \in G$, we have

$$\lim_{t\downarrow 0} \inf(1/t)(A(t)\, u(x) - u(x)) \leqslant 0 \leqslant \lim_{t\downarrow 0} \sup(1/t)(A(t)\, u(x) - u(x)),$$

where $A(t)$ is as in Section 2.1 with a kernel verifying (3.2). Then, u has no local strict extrema, unless u is constant. The proof is straightforward and will be omitted.

4. The Dirichlet Problem

4.1. In this section, we present our main results on the approximation of generalized solutions of elliptic second-order Dirichlet problems by means of solutions of certain integral equations.

The basic ingredients are: a uniformly elliptic differential operator of the form

$$E - \lambda \equiv \sum_{i,k=1}^{m} a_{ik}(x)\, \partial^2/\partial x_i\, \partial x_k - \lambda, \tag{4.1}$$

an open bounded set $G \subset R^m$, and a continuous map of the boundary ∂G of G into the reals (Dirichlet data). For convenience, we assume that the Dirichlet data are extended to a continuous map from the whole of R^m into the reals and we call φ such an extension. In (4.1), λ is a nonnegative constant parameter. In the case $\lambda = 0$ (the most interesting case, of course), the boundedness of the domain G is an essential restriction for our purposes, while such a restriction may be dropped in the case $\lambda > 0$. The ellipticity of (4.1) means that the matrix of the coefficients $a(x) = (a_{ik}(x))_{i,k=1,\ldots,m}$ is restricted to having its eigenvalues in some compact subset of $]0, +\infty[$ independent of x; compare with (2.11). We emphasize that, except in Theorem 4.3, no smoothness assumptions are made on the coefficients a_{ik}. They are supposed to be merely measurable and bounded functions (defined in the whole of R^m).

We consider solutions of the Dirichlet problem:

$$\begin{aligned} &\text{(a)} \quad Eu - \lambda u = f, \\ &\text{(b)} \quad u - \varphi\,|_{\partial G}\text{, the restriction of } u - \varphi \text{ to } \partial G, = 0, \end{aligned} \tag{4.2}$$

which are real-valued functions with the properties

(i) u is continuous on the closure $\bar{G}$ of G;

(ii) $Eu \in L^p(G)$ for some p $(1 \leqslant p \leqslant +\infty)$, in a generalized sense.

To explain condition (ii), we choose a kernel $P(x, \xi)$ having the properties (i)–(v) of Section 2.2 and in addition, the following one:

$$\operatorname{spt} P(x, \cdot) \equiv \text{the support of } R^m \ni \xi \to P(x, \xi) \subseteq \{\xi \in R^m : |\xi| \leqslant 1\}, \tag{4.3}$$

which guarantees that the average,

$$t^{-m/2} \int_{R^m} P(x, (x-y)/t^{1/2})\, u(y)\, dy, \tag{4.4}$$

of a function u (locally integrable in the domain G) is well defined for every x having distance $> t^{1/2}$ from the exterior of G, i.e., the average is a function on the set

$$G(t) = \{x \in G : \operatorname{distance}(x, \partial G) > t^{1/2}\}. \tag{4.5}$$

Perhaps the simplest example of such a kernel is the following (compare with Example 1, Section 2.3).

$$
\begin{aligned}
P(x, \xi) &= (m/\omega_m)(\det a(x))^{-1/2} && \text{if} \quad (a(x)^{-1}\,\xi, \xi) \leqslant 1 \\
&= 0 && \text{if} \quad (a(x)^{-1}\,\xi, \xi) > 1,
\end{aligned}
\tag{4.6}
$$

which verifies (4.3) under a suitable normalization of the matrix $a(x)$ (i.e., the largest eigenvalue of $a(x) = 1$). In the case (4.6), the average (4.4) is simply

$$
(1/\text{meas}\ \mathscr{E}(x, t^{1/2})) \int_{\mathscr{E}(x, t^{1/2})} u(y)\, dy, \tag{4.7a}
$$

the arithmetic mean on the ellipsoid

$$
\mathscr{E}(x, t) = \{y \in R^m : (a(x)^{-1}\,(y - x), y - x) < t\}. \tag{4.7b}
$$

For a function u locally integrable in G, we say $Eu \in L^p(G)$ if there exists a function belonging to $L^p(G)$, to be called Eu, such that

$$
\int_{G(t)} \left| Eu(x) - \frac{1}{Ct} \left(t^{-m/2} \int_{R^m} P\left(x, \frac{x-y}{t^{1/2}}\right) u(y)\, dy - u(x) \right) \right|^p dx \to 0\ (t \downarrow 0), \tag{4.8}
$$

where the integral is taken on the set (4.5) and C is the constant related to P and to the coefficients a_{ik} by (iv), Section 2.2. The usual modification must be performed in (4.8) when $p = \infty$.

We want to show that any solution u of the Dirichlet problem (4.2), verifying conditions (i) and (ii) with $p = \infty$ (if any such solution exists), can be approximated in the metric of $L^\infty(G)$ by means of solutions (which exist and are unique, as will be proved) of certain integral equations, to be described presently. As a corollary, we obtain uniqueness for the quoted solutions of the Dirichlet problem (4.2).

It should be mentioned that our method is similar to the argument of Wasow and Forsythe [68, Sections 23.2–23.4].

To motivate the integral equations we have in mind, we start from the differential equation (4.2a), $Eu(x) - \lambda u(x) = f(x)$, and we replace $Eu(x)$ at any point x of the set (4.5) with the differential quotient

$$
(1/Ct) \left(t^{-m/2} \int_{R^m} P(x, (x-y)/t^{1/2})\, u(y) dy - u(x) \right).
$$

Afterward, we extend the boundary condition (4.2b) to the whole of $G\backslash G(t)$. In this way, we obtain the pair of equations

$$
\begin{aligned}
&\text{(a)} \quad t^{-m/2} \int_{R^m} P(x, (x-y)/t^{1/2})\, v(t, y))\, dy \\
&\qquad = (1 + C\lambda t)\, v(t, x) + Ctf(x) \qquad \text{if} \quad x \in G(t), \\
&\text{(b)} \quad v(t, x) = \varphi(x) \qquad\qquad\qquad\qquad \text{if} \quad x \in G\backslash G(t),
\end{aligned}
\tag{4.10}
$$

where $v(t, x)$ is the new unknown. Then, we split the integral in (4.10a) into the sum

$$\int_{G(t)} \cdots\, dy + \int_{G\backslash G(t)} \cdots\, dy$$

and we insert (4.10b) into the second term. Therefore, the final form of our integral equation is

$$
\begin{aligned}
&t^{-m/2} \int_{G(t)} P(x, (x-y)/t^{1/2})\, v(t, y)\, dy \\
&\quad = (1 + C\lambda t)\, v(t, x) \\
&\qquad - t^{-m/2} \int_{G\backslash G(t)} P(x, (x-y)/t^{1/2})\, \varphi(y)\, dy + Ctf(x) \qquad \text{if} \quad x \in G(t) \\
&\qquad\qquad v(t, x) = \varphi(x) \qquad \text{if} \quad x \in G\backslash G(t).
\end{aligned}
\tag{4.11}
$$

Note that (4.11a) refers exclusively to the restriction of $v(t, \cdot)$ to the set $G(t)$, while the values of $v(t, \cdot)$ on the remaining part of G are given directly by (4.11b).

As will be clear from proofs, the above conditions on P (i.e., the (i)–(v) of Section 2.2 and (4.3)) are enough for a discussion of the integral equation (4.11a) in the case $\lambda > 0$. More sophisticated hypotheses, concerning supports, occur in the case $\lambda = 0$.

First, we suppose, besides (4.3), that the support of $P(x, \ )$ contains a ball, say:

$$\operatorname{spt} P(x, \cdot) \supseteq \{\xi \in R^m : |\xi| \leqslant r\} \qquad (0 < r \leqslant 1). \tag{4.12}$$

Second, we assume that for every pair of open sets A, B such that

$$\bar{B} \subseteq \{x \in R^m : 0 < \operatorname{meas} A \cap \operatorname{spt} P(x, (x - \cdot)/t^{1/2})\}, \tag{4.13a}$$

the function

$$x \to \int_A t^{-m/2} P(x, (x-y)/t^{1/2})\, dy$$

(which is obviously positive at every point of $\bar{B}$, by the definition of A and B) is uniformly positive on B, i.e.:

$$\operatorname*{ess\,inf}_{x \in B} \int_A t^{-m/2} P(x, (x-y)/t^{1/2})\, dy > 0. \tag{4.13b}$$

As a matter of a fact, a somewhat weaker condition than (4.13) will suffice. That is, we can replace (4.13a) by the less demanding condition

$$\bar{B} \text{ compact} \subseteq \{x \in R^m : \text{distance}(x, A) < rt^{1/2}\}, \tag{4.14}$$

where r is the number appearing in (4.12).

Clearly, the kernel (4.6) verifies (4.12) with r = the ellipticity constant μ. The same kernel verifies also (4.13) (at least in its weaker form). For

$$P(x, (x-y)/t^{1/2}) \geqslant (\text{a positive constant})\, \varphi((x-y)/\mu t^{1/2}),$$

where φ is the characteristic function of the unit ball and

$$x \to \int_A \varphi((x-y)/\mu t^{1/2})\, dy$$

is a continuous function, positive at any point of $\bar{B}$ (here A and B are as in (4.14)).

It is easy to check that all the kernels of Section 2.3 verify conditions (4.3), (4.12), and (4.13) (with a suitable choice of φ).

THEOREM 4.1. *For every $f \in L^\infty(G)$ and every $t > 0$, there exists exactly one solution $v(t, \cdot) \in L^\infty(G)$ of the system* (4.11). *Moreover, such a solution can be estimated as follows*:

$$\operatorname*{ess\,sup}_{x \in G} |v(t, x)|$$
$$\leqslant \max_{x \in \bar{G} \setminus G(t)} |\varphi(x)| + (\textit{const independent of } t, f, \varphi) \operatorname*{ess\,sup}_{x \in G(t)} |f(x)|. \tag{4.15}$$

THEOREM 4.2. *Let $f \in L^\infty(G)$ and let u be a solution of the Dirichlet problem* (4.2) *such that u is continuous in $\bar{G}$, $Eu \in L^\infty(G)$. Conclusion:*

$$\operatorname*{ess\,sup}_{x\in G} | u(x) - v(t,x)| \to 0 \qquad \text{as} \quad t \downarrow 0,$$

where $v(t, \cdot)$ is the $L^\infty(G)$-solution of (4.11).

COROLLARY. *The Dirichlet problem* (4.2) *has at most, one solution u, continuous in $\bar{G}$ and such that $Eu \in L^\infty(G)$.*

The corollary follows trivially from Theorem 4.2 and the uniqueness statement of Theorem 4.1.

Proof of Theorem 4.2. The difference $v(t, x) - u(x)$ is a solution of the system

$$\begin{aligned} t^{-m/2} \int_{G(t)} & P(x, (x-y)/t^{1/2})[v(t,y) - u(y)]\, dy \\ &= (1 + C\lambda t)[v(t,x) - u(x)] \\ &\quad + Ct \left[Eu(x) - \frac{1}{Ct}\left(t^{-m/2} \int_{R^m} P\left(x, \frac{x-y}{t^{1/2}}\right) u(y)\, dy - u(x)\right)\right] \\ &\quad - t^{-m/2} \int_{G\setminus G(t)} P\left(x, \frac{x-y}{t^{1/2}}\right) [\varphi(y) - u(y)]\, dy \qquad \text{if} \quad x \in G(t), \\ v(t,x) - u(x) &= \varphi(x) - u(x) \qquad \text{if} \quad x \in G\setminus G(t). \end{aligned}$$

By applying to this system the estimate (4.15) we get

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in G} & | v(t,x) - u(x)| \\ &\leqslant \max_{x\in\setminus G\bar{G}(t)} | \varphi(x) - u(x)| \\ &\quad + (\text{const}) \operatorname*{ess\,sup}_{x\in G(t)} \left| Eu(x) - \frac{1}{Ct}\left(t^{-m/2} \int_{R^m} P\left(x, \frac{x-y}{t^{1/2}}\right) u(y)\, dy - u(x)\right)\right|, \end{aligned}$$

from which the desired result follows.

THEOREM 4.3. *Assume that the coefficients a_{ik} are twice continuously* *differentiable with second derivatives bounded and uniformly continuous. Assume that* $f \in L^p(G)$ *for some* p $(1 \leqslant p < \infty)$ *and that*

$$\lambda > \frac{B}{p} = \frac{1}{p} \sup_{x \in R^m} \sum_{i,k=1}^{m} \frac{\partial^2 a_{ik}}{\partial x_i \, \partial x_k}(x).$$

Then, for every sufficiently small t, *the integral equation* (4.11) *has exactly one solution* $v(t, \cdot)$ *belonging to* $L^p(G)$, *and such a solution can be estimated as follows*:

$$\| v(t, \cdot)\|_{L^p(G)} \leqslant (\text{meas } G)^{1/p} \max_{x \in \bar{G} \setminus G(t)} |\varphi(x)| + \frac{1}{\lambda - (B/p) - \epsilon} \left(\int_{G(t)} |f(x)|^p \, dx\right)^{1/p},$$

where ϵ *is any a priori given number such that* $0 < \epsilon < \lambda - (B/p)$.

It is understood that the kernel P *must be choosen in such a way that the second derivatives of* P *with respect to* x *are uniformly continuous in* $R^m \times R^m$.

Moreover, let u *be a solution of the Dirichlet problem* (4.2) *such that* u *is continuous in* $\bar{G}$, $Eu \in L^p(G)$. *Then*:

$$\| u - v(t, \cdot)\|_{L^p(G)} \to 0, \qquad \text{if} \quad t \downarrow 0.$$

4.2. *Proof of Theorem* 4.1. The integral equation (4.11a) can be written

$$w(t, \cdot) = (1 + C\lambda t)^{-1} K(t) \, w(t, \cdot) + \frac{Ct}{1 + C\lambda t} z(t, \cdot), \tag{4.20}$$

where

$$w(t, \cdot) = v(t, \cdot) \,|_{G(t)}$$

is the restriction of the unknown $v(t, \cdot)$ to the set $G(t)$, z is given by

$$z(t, x) = -f(x) + \left(\frac{1}{Ct}\right) t^{-m/2} \int_{G \setminus G(t)} P\left(x, \frac{x - y}{t^{1/2}}\right) \varphi(y) \, dy, \tag{4.21}$$

and $K(t)$ is the operator

$$K(t) \, u(x) = t^{-m/2} \int_{G(t)} P(x, (x - y)/t^{1/2}) \, u(y) \, dy. \tag{4.22}$$

Keeping in mind the situation

$$P(x, \xi) \geqslant 0, \qquad t^{-m/2} \int_{R^m} P(x, (x-y)/t^{1/2})\, dy = 1,$$

we see that $z(t, \cdot)$ is a bounded function

$$| z(t, x)| \leqslant \text{ess sup } |f| + (1/Ct) \max |\varphi| \tag{4.23}$$

and that $K(t)$ is a contraction on $L^\infty(G(t))$

$$\operatorname*{ess\,sup}_{x \in G(t)} | Ku(x)| \leqslant \operatorname*{ess\,sup}_{x \in G(t)} | u(x)|. \tag{4.24}$$

The proof will be accomplished in two steps: (i) existence and uniqueness of the solution $w(t, \cdot) \in L^\infty(G(t))$ of the integral Eq. (4.20) for every fixed $t > 0$; (ii) estimates of such a solution. The results concerning w, together with (4.11b), will give the desired properties of v.

Step (i) must be covered in two ways, according to whether $\lambda > 0$ or $\lambda = 0$.

If $\lambda > 0$, the operator $(1 + C\lambda t)^{-1} K(t)$ appearing in (4.20) is such that

$$\|(1 + C\lambda t)^{-1} K(t)\| \leqslant 1/(1 + C\lambda t) = \text{a number strictly less than } 1, \tag{4.25}$$

where $\| \ \|$ is the norm in the algebra of bounded linear operators in $L^\infty(G(t))$. Formula (4.25) is an obvious consequence of (4.24). The conclusion follows at once from (4.23) and (4.25). Note that the boundedness of the domain G does not occur here.

If $\lambda = 0$, we achieve the result from the following basic lemma, whose proof we defer until later.

LEMMA 4.4. *The operator $K(t)$ is strictly contractive on $L^\infty(G(t))$ (i.e., has norm <1) if the diameter of G is sufficiently small. If G is any bounded domain, a power $K(t)^n$ of $K(t)$ (with a sufficiently large n, depending on t and on the diameter of G) is strictly contractive on $L^\infty(G(t))$.*

Continuing with the proof of Theorem 4.1, we next discuss step (ii): estimation of the solution w.

From our previous work, we infer the representation of w in a Neuman series:

$$w(t, \cdot) = (Ct/(1 + C\lambda t)) \sum_{n=0}^{+\infty} (1 + C\lambda t)^{-n} K(t)^n z(t, \cdot) \tag{4.26}$$

which is absolutely convergent in the metric of $L^\infty(G(t))$. The powers $K(t)^n$ $(n = 1, 2, 3,...)$ are integral operators

$$K(t)^n u(x) = \int_{G(t)} k_n(t; x, y) u(y) dy, \tag{4.27}$$

whose kernels are

$$k_1(t; x, y) = t^{-m/2} P\left(x, \frac{x-y}{t^{1/2}}\right),$$

$$k_n(t; x, y) = \int_{G(t)} k_{n-1}(t; x, \xi) t^{-m/2} P\left(\xi, \frac{\xi - y}{t^{1/2}}\right) d\xi, \tag{4.28}$$

$$k_n(t; x, y) \geqslant 0, \tag{4.29}$$

$$\| K(t)^n \| = \operatorname*{ess\,sup}_{x\in G(t)} \int_{G(t)} k_n(t; x, y) dy, \tag{4.30}$$

where $\| \ \|$ is, as before, the norm of the bounded linear operators in $L^\infty(G(t))$.

From (4.24), or Lemma 4.4 and Eq. (4.30), it follows that the series

$$\sum_{n=1}^{+\infty} (1 + C\lambda t)^{-n} k_n(t; x, y) \tag{4.31a}$$

converges to a kernel $k(t; x, y) \geqslant 0$ such that

$$\operatorname*{ess\,sup}_{x\in G(t)} \int_{G(t)} k(t; x, y) dy < \infty, \tag{4.31b}$$

the convergence being of the type

$$\operatorname*{ess\,sup}_{x\in G(t)} \int_{G(t)} \left| \sum_{n=1}^{N} (1 + C\lambda t)^{-n} k_n(t; x, y) - k(t; x, y) \right| dy \to 0,$$

$$\text{if } N \to +\infty. \tag{4.31c}$$

Incidentally, the recurrence relation (4.28), coupled with (4.30), implies that the convergence of (4.31) is uniform in $G(t) \times G(t)$, if P is bounded.

Thus, formula (4.26) can be written, using (4.21), as

$$w(t, x) = \int_{G \setminus G(t)} k(t; x, y)\, \varphi(y)\, dy$$
$$- (Ct/(1 + C\lambda t)) \left[f(x) + \int_{G(t)} k(t; x, y)\, f(y)\, dy \right]. \tag{4.32}$$

We emphasize that

$$k(t; x, y) \geqslant 0. \tag{4.34}$$

Now, we estimate

$$\int_{G \setminus G(t)} k(t; x, y)\, dy, \tag{4.35}$$
$$(x \in G(t)),$$
$$(Ct/(1 + C\lambda t)) \left[1 + \int_{G(t)} k(t; x, y)\, dy \right]. \tag{4.36}$$

To do this, we remark that if $f(x) \equiv -\lambda$ and $\varphi(x) \equiv 1$, the solution of (4.20) is $w(t, x) \equiv 1$. This is very easy to check. Thus, from (4.32), we have

$$1 = \lambda \times \text{the function (4.36)} + \text{the function (4.35)}.$$

Hence, as both terms are nonnegative:

$$\int_{G \setminus G(t)} k(t; x, y)\, dy \leqslant 1 \qquad (=1 \text{ if } \lambda = 0). \tag{4.37}$$

Also, we obtain the function (4.36) $\leqslant 1/\lambda$ (only if $\lambda > 0$).

If λ may be zero, we must proceed differently in order to estimate (4.36). Choosing $\varphi(x) = k - |x|^2$ and $f(x) = -2 \sum_{i=1}^{m} a_{ii}(x) - \lambda(k - |x|^2)$ (where k is a constant and a_{ii} are coefficients of the differential operator E), the solution of (4.20) is $w(t, x) = k - |x|^2$. This is easy to check using the properties (i)–(iv), Section 2.2, of the kernel P. Fix k so large that $k - |x|^2 \geqslant 0$ at any point $x \in \bar{G}$ and insert in (4.32) the φ, f, w

specified above. By trivial considerations, one obtains the function (4.36) $\times$ 2 $\inf \sum_{i=1}^{m} a_{ii} \leqslant k$, that is,

$$\frac{Ct}{1+C\lambda t}\left[1+\int_{G(t)} k(t; x, y)\, dy\right]$$
$$\leqslant \text{(const depending on the diameter of } G \text{ only)}/\text{(the ellipticity const } \mu). \tag{4.38}$$

Putting together (4.32), (4.34), (4.37) and (4.38), we obtain

$$\operatorname*{ess\,sup}_{x\in G(t)} |w(t, \;)| \leqslant \max_{x\in G\setminus G(t)} |\varphi(x)| + (\text{const}) \operatorname*{ess\,sup}_{x\in G(t)} |f(x)|,$$

where the constant can be estimated by $1/\lambda$ if $\lambda > 0$, or by the right side of (4.38) if $\lambda = 0$. The proof is complete.

Proof of Lemma 4.4. Denoting $\| \;\|$ as the norm of the bounded linear operators in $L^\infty(G(t))$, we have, as noted above (see (4.30)),

$$\| K(t)\| = \operatorname*{ess\,sup}_{x\in G(t)} \int_{G(t)} t^{-m/2}P(x, (x-y)/t^{1/2})\, dy. \tag{4.40}$$

Since

$$\int_{R^m} t^{-m/2}P(x, (x-y)/t^{1/2})\, dy = 1,$$

we have

$$\| K(t)\| = 1 - \operatorname*{ess\,inf}_{x\in G(t)} \int_{R^m\setminus G(t)} t^{-m/2}P(x, (x-y)/t^{1/2})\, dy. \tag{4.41}$$

Let the diameter d of G be so small that

$$d < rt^{1/2}, \tag{4.42a}$$

where r is the number appearing in (4.12). We claim

$$\operatorname*{ess\,inf}_{x\in R^m} \int_{R^m\setminus G(t)} t^{-m/2}P(x, (x-y)/t^{-1/2})\, dy > 0. \tag{4.42b}$$

In fact, if $d < rt^{1/2}$, we also have diam $G(t) < rt^{1/2}$. Hence,

$$\overline{G(t)} \subset B(x, r(t^{1/2})), \qquad \text{for every} \quad x \in \overline{G(t)},$$

where $B(x, r)$ is the open ball with center at x and radius r. By the hypothesis (4.12), we have

$$\operatorname{spt} P(x, (x - \cdot)/t^{1/2}) \supseteq \overline{B(x, rt^{1/2})}. \tag{4.43}$$

Consequently,

$$\begin{aligned}
&[R^m \backslash \overline{G(t)}] \cap \operatorname{spt} P(x, (x - \cdot)/t^{1/2}) \\
&\quad \supset [R^m \backslash \overline{G(t)}] \cap B(x, rt^{1/2}) \begin{cases} \text{trivially open nonvoid if } x \notin \overline{G(t)} \\ = B(x, rt^{1/2}) \backslash \overline{G(t)}, \qquad \text{if} \quad x \in \overline{G(t)} \end{cases} \\
&\quad = \text{an open nonvoid set for every } x \in R^m.
\end{aligned}$$

Thus, (4.42) follows from the hypothesis (4.13). Hence, we have proved $\| K(t)\| < 1$ if the diameter of G verifies (4.42a).

Now, let d be arbitrarily large and let n be any integer such that

$$d/n < rt^{1/2}, \tag{4.44a}$$

r having the same meaning as before. We shall prove that

$$\| K(t)^n \| < 1. \tag{4.44b}$$

To do this, consider the kernels $a_n(t; x, y)$ of the powers $A(t)^n$ of the operator $A(t)$ (2.12). Such kernels are defined by the recurrence relations

$$\begin{aligned}
a_1(t; x, y) &= t^{-m/2} P(x, (x - y)/t^{1/2}), \\
a_n(t; x, y) &= \int_{R^m} a_{n-1}(t, x; \xi)\, t^{-m/2} P(\xi, (\xi - y)/t^{1/2})\, d\xi.
\end{aligned} \tag{4.45}$$

If we call $k_n(t; x, y)$ the kernel of $K(t)^n$, an easy inspection of (4.28) and (4.45) shows that

$$k_n(t; x, y) \leqslant a_n(t; x, y). \tag{4.46}$$

Thus, (4.30) implies that

$$\| K(t)^n \| \leqslant \operatorname*{ess\,sup}_{x \in G(t)} \int_{G(t)} a_n(t; x, y)\, dy. \tag{4.47}$$

Since

$$\int_{R^m} a_n(t; x, y)\, dy = 1 \qquad (n = 1, 2, \ldots),$$

we have the formula

$$\| K(t)^n \| \leqslant 1 - \operatorname*{ess\,inf}_{x \in G^m} \int_{R^m \setminus G(t)} a_n(t; x, y)\, dy. \tag{4.49}$$

Let $\{U_n\}$ be any sequence of open sets such that

$$U_0 = \text{the empty set}, \qquad \operatorname{diam} U_1 < rt^{1/2}. \tag{4.50a}$$

We suppose that the sequence increases,

$$\bar{U}_{n-1} \subset U_n ,$$

but not too fast, in the sense that

$$U_{n-1} \supset \{x \in U_n : \text{distance}(x, R^m \setminus U_n) \geqslant rt^{1/2}\}. \tag{4.50b}$$

We claim that

$$\operatorname*{ess\,inf}_{x \in R^m} \int_{R^m \setminus U_n} a_n(t; x, y)\, dy > 0 \qquad (n = 1, 2, ...). \tag{4.50c}$$

Obviously, (4.49) and (4.50) imply (4.44).

To prove (4.50) let us call p_n the left side of (4.50c). Since

$$p_1 = \operatorname*{ess\,inf}_{x \in R^m} \int_{R^m \setminus U_1} t^{-m/2} P(x, (x - y)/t^{1/2})\, dy$$

and $\operatorname{diam} U_1 < rt^{1/2}$, we have $p_1 > 0$ as in the proof of (4.42). Analogously, (4.43), (4.50b), and the hypothesis (4.13) yield

$$q_n \equiv \operatorname*{ess\,inf}_{x \in R^m \setminus U_{n-1}} \int_{R^m \setminus U_n} t^{-m/2} P(x, (x - y)/t^{1/2})\, dy > 0.$$

On the other hand, the recurrence relations (4.45) give

$$\begin{aligned} p_n &= \operatorname*{ess\,inf}_{x \in R^m} \int_{R^m} a_{n-1}(t; x, \xi)\, d\xi \int_{R^m \setminus U_n} t^{-m/2} P(\xi, (\xi - y)/t^{1/2})\, dy \\ &\geqslant \operatorname*{ess\,inf}_{x \in R^m} \int_{R^m \setminus U_{n-1}} \cdots d\xi \times \operatorname*{ess\,inf}_{\xi \in R^m \setminus U_{n-1}} \cdots . \end{aligned}$$

That is,

$$p_n \geqslant q_n p_{n-1} \qquad (n = 2, 3, 4 ...).$$

Hence,

$$p_n \geqslant q_n q_{n-1} \cdots q_2 p_1 = q_n q_{n-1} \cdots q_2 q_1 > 0.$$

The proof is complete.

Proof of Theorem 4.3 (sketched only). The solution v of (4.11) can be split into the sum $v = v_1 + v_2$, where v_1 is the solution of a system like (4.11) with f replaced by zero and v_2 is the solution of an analogous system with φ replaced by zero. The analysis of v_1 is covered by Theorem 4.1. The restriction $w(t, \cdot)$ of $v_2(t, \cdot)$ to the set $G(t)$ is determined by the integral equation

$$w(t, \cdot) = (1 + C\lambda t)^{-1} K(t)\, w(t, \cdot) - (Ct/(1 + C\lambda t))f,$$

where $K(t)$ is the operator (4.22). The arguments of Section 2.2 easily yield

$$\| K(t)u \|_{L^p(G(t))} \leqslant n(t)^{1/p} \| u \|_{L^p(G(t))},$$

where $n(t)$ is given by (2.15); so:

$$\|(1 + C\lambda t)^{-1} K(t)\| \qquad (= \text{the norm of} \cdots \text{in the ring of linear bounded operators in } L^p(G(t)))$$

$$\leqslant \frac{n(t)^{1/p}}{1 + C\lambda t} = 1 - \frac{Ct}{1 + C\lambda t}\left(\lambda - \frac{n(t)^{1/p} - 1}{Ct}\right)$$

$$\leqslant 1 - \frac{Ct}{1 + C\lambda t}\left(\lambda - \frac{n'(0)}{Cp} - \epsilon\right)$$

$$= 1 - \frac{Ct}{1 + C\lambda t}\left(\lambda - \frac{B}{p} - \epsilon\right)$$

(if $\epsilon > 0$ and t is small enough). Use has been made of Lemma 2.5. Then, if $\lambda > (B/p) + \epsilon$, $(1 + C\lambda t)^{-1} K(t)$ is strictly contractive in $L^p(G(t))$ and the norm of the resolvent operator $[1 + (1 + C\lambda t)^{-1} K(t)]^{-1}$ does not exceed $(1 + C\lambda t)/Ct(\lambda - (B/p) - \epsilon)$, etc.

4.3. *Remarks on the Green's Function*

For the sake of simplicity, we suppose that the Dirichlet datum φ is zero and we confine ourselves to the case $\lambda = 0$.

The solution $v(t, x)$ of the integral equation (4.11) can be written

$$v(t, x) = -Ctf(x)\, \mathbb{1}_{G(t)}(x) - \int_G g(t; x, y) f(y)\, dy, \tag{4.53}$$

where $\mathbb{1}_{G(t)}$ is the characteristic function of the set $G(t)$,

$$\begin{aligned} g(t; x, y) &= 0 && \text{if} \quad x \in G\backslash G(t), \quad \text{or} \quad y \in G\backslash G(t) \\ &= Ctk(t; x, y) && \text{if} \quad x \in G(t), \quad \text{and} \quad y \in G(t), \end{aligned} \tag{4.54}$$

and $k(t; x, y)$ is the sum of the series (4.31a).

From the proof of Theorem 4.1 (see (4.34) and (4.38)), we infer that

$$\begin{aligned} & g(t; x, y) \geqslant 0, \\ & \int_G g(t; x, y)\, dy \leqslant \text{constant independent of } t \text{ and } x. \end{aligned} \tag{4.45}$$

Formulas (4.55) have some consequences via Alaoglu's theorem (on the weak-$*$ compactness of closed balls of adjoint Banach spaces, see, e.g., [16, Theorem V 4.2]) and the Riesz theorem on the representation of bounded linear functionals on spaces of continuous functions. In fact, there exists a sequence $t_n \downarrow 0$ such that

$$\int_G g(t_n; x, y) f(y)\, dy \to \int_G f(y)\, \mu(x, dy)$$

for every $f \in C^0(\bar{G})$ and every $x \in \bar{G}$, where $\mu(x, \cdot)$ is some Radon measure on Borel subsets of $\bar{G}$.

If the coefficients a_{ik} of our differential operator E (4.1) are Lipschitz-continuous and the boundary of the domain G is sufficiently smooth, we can prove that the full family of kernels $g(t; x, y)$ converges (as t approaches zero) to the Green's function $g(x, y)$ of the differential operator, associated with the domain G. This remark is the main result of the present section. The above convergence must be understood in the following sense:

$$\int_G g(t; x, y) f(y)\, dy \to \int_G g(x, y) f(y)\, dy \tag{4.56}$$

uniformly in x as $t \downarrow 0$, for every Lipschitz-continuous function f. Of course, uniform convergence means here convergence in $L^\infty(G)$.

In the present context, we may give a very simple definition of the Green's function. To do this, we remark that in the present hypotheses, the Dirichlet problem

$$\begin{aligned} Eu(x) &= f(x), && \text{if} \quad x \in G, \\ u(x) &= 0, && \text{if} \quad x \in \partial G, \end{aligned} \tag{4.57}$$

has exactly one twice continuously differentiable solution u for every Lipschitz-continuous f (a corollary of the classical Schauder–Caccioppoli results; see [44]). On the other hand, such a solution u can be estimated [2–5, 49], by

$$\max |u| \leqslant (\text{const independent of } f) \|f\|_{L^m(G)}$$

Thus, the evaluation at a point $x \in \bar{G}$ of the $C^2(G)$-solution u of (4.57) is a functional of the right-hand side f, densely defined and bounded in $L^m(G)$. Hence, via the representation theorem on linear functionals on Lebesgue spaces, there exists exactly one function g on $G \times G$, such that

$$\begin{gathered} g(x, \cdot) \in L^{m'}(G), \\ u(x) = -\int_G g(x, y) f(y)\, dy \qquad \text{for every} \quad x \in \bar{G}, \end{gathered} \tag{4.58}$$

where $m' = m/(m-1)$, f is any Lipschitz-continuous function, and u is the $C^2(G)$-solution of (4.57). The g is the Green's function. We mention that this procedure can also be found in [29].

Formulas (4.53)–(4.58) and Theorems 2.1, 4.2, give the desired result (4.56).

5. Relations with Monte Carlo Methods

As is well known (see, e.g., [26]), the classical Monte Carlo method in elliptic boundary value problems consists essentially of the following two steps: (i) discretization of the problem (i.e., to find finite-difference problems whose solutions converge in some sense to those of the given differential problem); (ii) solution of the discretized problems by means of random walks.

The treatment of Dirichlet problems, described in Section 4, somewhat resembles Monte Carlo methods. For the sake of simplicity, we consider the Dirichlet problem

$$\begin{aligned} Eu &= 0 && \text{in } G, \\ u &= \varphi && \text{on the boundary of } G, \end{aligned} \tag{5.1}$$

where E, G, φ are as in Section 4.

First, Theorem 4.2 shows that the solution u of (5.1) (provided that this solution exists and belongs to a suitable functional class) is approximated in the topology of the uniform convergence

$$\operatorname*{ess\,sup}_{x \in G} | u(x) - v(t, x)| \to 0, \qquad (t \downarrow 0) \tag{5.2}$$

by the (bounded) solutions $v(t, \cdot)$ of certain integral equations, i.e., Eqs. (4.10) or (4.11) (with $\lambda = 0$ and f replaced by zero). As pointed out in the previous section, such integral equations simulate, in some sense, the differential problem (5.1).

Second, we can show that $v(t, x)$, the value of $v(t, \cdot)$ at any point x of G having distance $> t^{1/2}$ from the boundary ∂G, has an unbiased estimator that is related to a set of random walks starting at x.

The probabilistic interpretation of the solutions $v(t, \cdot)$ is quite simple if the kernel P, appearing in the relevant integral equations, has the form (4.6). Let us restrict ourselves to this case.

According to the proof of Theorem 4.1, (cf. (4.32)), the formula

$$v(t, x) = \int_{G \backslash G(t)} k(t; x, y)\, \varphi(y)\, dy, \qquad \text{for every} \quad x \in G(t). \tag{5.3}$$

holds. Here, $G(t)$ is the set (4.5),

$$G \backslash G(t) = \{x \in G : \text{distance}(x, \partial G) \leqslant t^{1/2}\},$$

$k(t; x, y)$ is the sum of the series (4.31) (with $\lambda = 0$), i.e.,

$$k(t; x, y) = \sum_{n=1}^{+\infty} k_n(t; x, y), \tag{5.4}$$

where $k_n(t; x, y)$ are the iterated kernels (4.28).

Consider random walks $\omega = (x_0, x_1, \ldots, x_n)$, such that

(i) the starting point x_0 is any given point x in the set $G(t)$;

(ii) every point x_k $(k = 1, 2, \ldots, n)$ is chosen randomly, with uniform probability distribution, in the ellipsoid

$$\mathscr{E}(x_{k-1}, t^{1/2})$$

(for the definition of such ellipsoid cf. (4.7b));

(iii) none of the points $x_1, \ldots, x_{n-1}$ can leave the set $G(t)$;

(iv) the terminating point x_n is in $G \backslash G(t)$.

Define a random variable X on the relevant sample space by the formula

$$X(\omega) = \text{the value of } \varphi \text{ at the terminating point of } \omega.$$

Formula (5.4) shows plainly that

$$\int_{G \setminus G(t)} k(t; x, y)\, \varphi(y)\, dy$$

is the mean value of X. Thus, by formula (5.3), the random variable X is an unbiased estimator of $v(t, x)$. Q.E.D.

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